Alexander Ziwet

AN INTRODUCTION

TO

PROJECTIVE GEOMETRY

AND ITS

APPLICATIONS

AN ANALYTIC AND SYNTHETIC TREATMENT

BY

ARNOLD EMCH, PH.D.,
Professor of Graphics and Mathematics in the University of Colorado

FIRST EDITION
FIRST THOUSAND

NEW YORK
JOHN WILEY & SONS
LONDON: CHAPMAN & HALL, LIMITED
1905

ROBERT DRUMMOND, PRINTER, NEW YORK.

PREFACE.

TREATISES on Projective Geometry are usually written with the object of presenting this science in a purely systematic form; hardly any attention is paid to the applications. As a rule the methods of "arithmetized" mathematics have to be transformed, made more concrete, before they lend themselves to the solution of practical problems; and this, in the judgment of many, disfigures the purely scientific method.

In this respect, projective geometry, geometry of position, is no exception. The puristic tendencies of von Staudt, Reye, and others culminate in the modern Italian school of geometer-logicians, headed by VERONESE [1] and ENRIQUES. The latter's projective geometry [2] contains an admirable logical presentation of the subject. With Enriques projective geometry is a "visual" science, and everything is foreign to it which cannot be based upon the axioms of vision.

It seems doubtful whether the axioms of vision alone can establish a sound projective geometry. Enriques himself, in his book, lets the fundamental elements of the first order be generated by motion! In this visual geometry metrical properties, which are indispensable in the applications, appear as special cases and are of secondary importance. Conics result from the theory of polarity.

On the other hand, FIEDLER, WIENER and others show that the methods of Poncelet, Steiner, Chasles, and Cremona naturally

[1] *Grundzüge der Geometrie*, Teubner, Leipzig.
[2] German translation, Teubner, Leipzig.

follow from the study of descriptive geometry. With them, projective and descriptive geometry are organically related and each branch benefits by its connection with the other. Little attention is paid to the so-called foundations.

As the present book has been written with a utilitarian purpose, considerable space is given to the applications; and in their treatment use has sometimes been made of original analytic and geometric methods of attack and solution. It has thus been found possible to include some new subject-matter and especially certain parts of modern analytical geometry.

In addition to the traditional contents of the standard elementary treatises, two chapters on pencils and ranges of conics, including cubics, and on the applications in mechanics have been added. The Steinerian transformation contained in Chapter IV, in connection with the study of plane cubics, is a brilliant example of the original power of projective geometry; and as it is elementary, it seems natural to introduce it after the theory of conics. As a novel feature the realization of collineations by linkages, described in Chapter V, may be mentioned.

Much time may profitably be devoted to the original problems and to the constructions involved in them. No first study of projective geometry can be successful without the constant use of ruler and compass.

My thanks are due to my colleague, Professor Ira M. De Long, for many valuable suggestions as to matters of form.

Corrections and suggestions as to either the form or the matter of the text are earnestly solicited.

ARNOLD EMCH.

BOULDER, COLORADO,
July, 1904.

CONTENTS.

CHAPTER I.

General Considerations. Anharmonic Ratio. Projective Ranges and Pencils. Polar Involution of the Circle.

CHAPTER II.

Collineation.

CHAPTER III.

Theory of Conics.

CHAPTER IV.

Pencils and Ranges of Conics. The Steinerian Transformation. Cubics.

CHAPTER V.

Applications in Mechanics.

PROJECTIVE GEOMETRY.

CHAPTER I.

GENERAL CONSIDERATIONS. ANHARMONIC RATIO. PROJECTIVE RANGES AND PENCILS. POLAR INVOLUTION OF THE CIRCLE.

§ 1. Geometric Quantities and their Signs.

Geometric quantities can be represented by numbers by assuming an arbitrary geometric quantity of the same kind as a unit.[1] To show this for linear quantities, assume any line AZ, Fig. 1, and a unit u. Measure off on AZ as many units u as

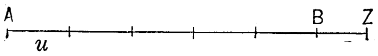


FIG. 1.

possible, so that the remainder $BZ < u$. Suppose that the number of units measured on AZ is a, so that $AZ = au + BZ$. Now consider BZ as a unit and u as the quantity to be measured. Suppose that BZ is contained b times in u and that the remainder $r < BZ$. Then $u = b \cdot BZ + r$. In a similar manner, consider r as a unit and BZ as the quantity to be measured. Suppose that r is contained c times in BZ and that the remainder is s, so

[1] See LAGRANGE'S *Lectures on Elementary Mathematics* (translated by Th. J. McCormack, Open Court Publ. Comp., Chicago), p. 3.

$$\frac{\partial V}{\partial b} = w - b\int \frac{dr}{r^2\sqrt{2f(r)+2h-\frac{b^2}{r^2}}} = b',$$

$$\frac{\partial V}{\partial \alpha} = \frac{b[\sin\alpha\cos\eta - \cos\alpha\sin\eta\cos(\vartheta-\beta)]}{\sin w} = \alpha',$$

$$\frac{\partial V}{\partial \beta} = \frac{-b\sin\alpha\sin\eta\sin(\vartheta-\beta)}{\sin w} = \beta',$$

wo w durch die Gleichung

$$\cos w = \cos\alpha\cos\eta + \sin\alpha\sin\eta\cos(\vartheta-\beta)$$

bestimmt ist, sind dann aber die eingeführten sechs Constanten nicht unabhängig von einander, sondern es findet unter ihnen, in Folge der identischen Gleichung

$$\left(\frac{\partial V}{\partial \alpha}\right)^2 + \frac{1}{\sin^2\alpha}\left(\frac{\partial V}{\partial \beta}\right)^2 = b^2,$$

die Relation.

$$\alpha'\alpha' + \frac{\beta'\beta'}{\sin^2\alpha} = b^2$$

statt, und es vertreten demnach die beiden Gleichungen

$$\frac{\partial V}{\partial \alpha} = \alpha', \quad \frac{\partial V}{\partial \beta} = \beta'$$

nur die Stelle von einer.

Die Zeit erhält man durch die Gleichung:

$$t+\tau = \frac{\partial V}{\partial h} = \int \frac{dr}{\sqrt{2f(r)+2h-\frac{b^2}{r^2}}}.$$

Die Integrale erster Ordnung werden:

$$\frac{\partial V}{\partial x} = \frac{dx}{dt} = \frac{-b(\cos\alpha - \cos\eta\cos w)}{r\sin w} + \cos\eta\sqrt{2f(r)+2h-\frac{b^2}{r^2}},$$

$$\frac{\partial V}{\partial y} = \frac{dy}{dt} = \frac{-b(\sin\alpha\cos\beta - \sin\eta\cos\vartheta\cos w)}{r\sin w} + \sin\eta\cos\vartheta\sqrt{2f(r)+2h-\frac{b^2}{r^2}},$$

$$\frac{\partial V}{\partial z} = \frac{dz}{dt} = \frac{-b(\sin\alpha\sin\beta - \sin\eta\sin\vartheta\cos w)}{r\sin w} + \sin\eta\sin\vartheta\sqrt{2f(r)+2h-\frac{b^2}{r^2}}.$$

Führt man statt der Differentiale der rechtwinkligen Coordinaten die Differentiale der Polarcoordinaten ein, so erhält man hieraus:

the left to the right. Increase and decrease of geometric quantity on this line are measured by the amount of displacement of a moving point, or by the length of the line between the original and final position of a moving point on this line. The formal laws of all displacements on this line are those of the group. Thus,

$$AB+BC=AC \tag{1}$$

shows that two displacements succeeding each other are equivalent to a single displacement of the same kind and of the same system (group).[1] It follows further that

$$AB+BC+CA=0;^{2} \tag{2}$$

hence by substitution of (1) in (2)

$$AC+CA=0,$$

or

$$CA=-AC;$$

i.e., two displacements, or geometric quantities, which are described in opposite directions are of opposite sign. The same conclusions are reached when angular displacements are considered. It is a universal convention to designate all geometrical quantities which are obtained by displacements on a line from the left to the right as positive and those in the opposite direction as negative. In a similar manner, angles formed by angular displacements counter-clockwise are assumed as positive and those clockwise as negative. The conception of the group is general and also comprises the determination of such geometric quantities as areas and volumes.

In case of a surface assume a pole O and any line l on this surface. A point P is moving on l, and in any position of its motion is connected to the point O by a geodesic of the surface. The generalized radius vector OP then sweeps over a certain area

---

[1] It is beyond the limits of this book to enter into a discussion of groups in this connection.

[2] Möbius: *Barycentrische Calcul*, § 1.

which is subject to the laws of the group. Thus if P moves from A to B to C, counter-clockwise with respect to O,

$$OAB+OBC=OAC,$$

$$OAB+OBC+OCA=0;$$

hence
$$OAC+OCA=0,$$

or
$$OCA=-OAC.$$

Two areas have therefore to be considered as of opposite sign if their boundaries are described in opposite senses. According to the distinction which we have made, an area is positive or negative according as its boundary is described counter-clockwise or clockwise, respectively. From this it follows further that if the point P describes a closed line on a surface, the radius vector OP sweeps over an area equal to the area enclosed by the curve. For a triangle ABC, Fig. 2,

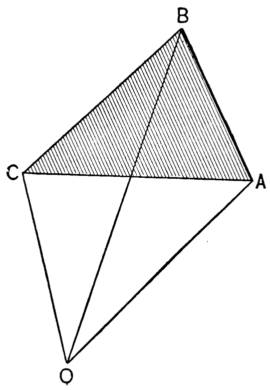


FIG. 2.

$$ABC=OAB+OBC+OCA,$$

where OCA is negative. On the other hand

$$CBA=OCB+OBA+OAC$$
$$=-OBC-OAB+OAC;$$

hence
$$ABC+CBA=0,$$
$$CBA=-ABC.$$

The same reasoning may be extended to the determination of volumes, which is left to the reader as an exercise.

Ex. 1. If A, B, C, D are four collinear points, prove that

$$BC\cdot AD+CA\cdot BD+AB\cdot CD=0.$$

Ex. 2. For the same points prove

$$DA^2 \cdot BC + DB^2 \cdot CA + DC^2 \cdot AB = -BC \cdot CA \cdot AB.$$

§ 2. Anharmonic Ratio.[1] Projective Transformation of the Points of a Straight Line.

1. *Critical Note.*—VON STAUDT in his classical works [2] on the geometry of position created a system with the principal purpose of laying the foundations of geometry without the aid of metrical considerations. He introduced the word "Wurf" as an equivalent of anharmonic ratio and attached to it a meaning independent of any ratio. The anharmonic ratio is considered as a property of the "Wurf", so that, according to v. Staudt, metrical geometry is based upon projective geometry, or rather the geometry of position. STEINER, on the contrary, took the anharmonic ratio as a starting-point in his investigations.[3] In a recent paper [4] POINCARÉ has pointed out "that from a certain point of view the geometry of v. Staudt is predominantly a visual geometry, while that of Euclid is predominantly muscular." In other words, the two geometries are derived from experiences in optics and kinematics, respectively.

In works with practical purposes, where applications form an important part, it is probably of the greatest advantage to take one view or the other according to the simplicity of the treatment which it may afford.

This method, although objectionable from the standpoint of pure geometry, reflects the development of geometric science itself.

---

[1] I shall use the expression *anharmonic ratio*, because it is used by the translators of Reye's and Cremona's treatises on projective geometry and by a majority of English authors. *Double ratio*, corresponding to the German *Doppelverhältnis*, is presumably a better designation.

[2] *Geometrie der Lage*, 1847. *Beiträge*, 1856–60.

[3] *Systematische Entwickelung der Abhängigkeit geometrischer Gestalten*, etc., 1832.

[4] *On the Foundations of Geometry*, Monist, No. 1, Vol. IX.

2. The anharmonic ratio of four points A, B, C, D on a line,

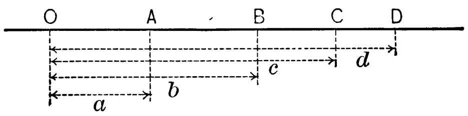


FIG. 3.

a straight line for the sake of simplicity, where (A, B) shall be designated as the first, (C, D) as the second pair, Fig. 3, is

$$\frac{AC}{BC} \Big/ \frac{AD}{BD} = k. \tag{1}$$

As AC, BC, AD, BD are all positive quantities, k will be a positive number. It is clear that this is not the only anharmonic ratio that may be formed between the four points. As there are 24 permutations possible between four elements, there will also be 24 anharmonic ratios. Some of these, however, have the same value, and it may easily be verified that there are only 6 different anharmonic ratios possible. Designating (1) by the symbol $(ABCD)$,[1] these are

$$\begin{cases} (ABCD) = k, \\ (BACD) = \dfrac{1}{k}, \\ (BCAD) = \dfrac{1-k}{k}, \\ (CBAD) = \dfrac{k}{1-k}, \\ (CABD) = \dfrac{1}{1-k}, \\ (ACBD) = 1-k. \end{cases} \tag{2}$$

If the points A, B, C, D are located by their displacements a, b,

[1] MÖBIUS, *Barycentrische Calcul*, § 183.

c, d from a fixed point O, the first anharmonic ratio assumes the form

(3) $$\frac{c-a}{c-b} \Big/ \frac{d-a}{d-b} = k.$$

3. This expression leads to the solution of the important problem to find all pairs of points, X, Y, which with two fixed points A and B form the constant anharmonic ratio k. Associating with X and Y the displacements x and y from O, the condition, according to (3), is

(4) $$\frac{x-a}{x-b} = k\frac{y-a}{y-b},$$

or, solved for x,

(5) $$x = \frac{(a-bk)y - ab(1-k)}{(1-k)y - (b-ak)}.$$

From this it is seen that to every value of y corresponds one and only one value of x satisfying the condition of the problem, and *vice versa*. Taking any four points Y_1, Y_2, Y_3, Y_4, and determining the corresponding points X_1, X_2, X_3, X_4 according to (5), there is found the relation

$$(X_1X_2X_3X_4) = (Y_1Y_2Y_3Y_4);$$

i.e., any four points of the series (X) and their corresponding points of the series (Y) satisfying the condition (5) have the same anharmonic ratio. Two series or ranges of points with this property are said to be *projective*. Formula (5) is the analytical expression for these projective ranges of points; it effects a *projective transformation* [1] of the points of a straight line.

For $y=a$, $x=a$, and for $y=b$, $x=b$; i.e., the transformation leaves the points A and B invariant; they are called the *double-*

[1] The word *projective* was first used by Poncelet in his great work: *Traité des propriétés projectives des figures*, 1822. Möbius was the first who gave an *analytical* representation of projective transformations, in *Der barycentrische Calcul*, 1827.

points of the transformation, or of the projective ranges of points. From (4) follows immediately that *every pair of corresponding points forms a constant anharmonic ratio with the double-points.*

On the other hand every transformation of the form

$$x=\frac{Ay+B}{Cy+D} \tag{6}$$

is projective. To prove this assume four points Y_1, Y_2, Y_3, Y_4, and determine the corresponding points X_1, X_2, X_3, X_4. Let y_1, y_2, y_3, y_4 and x_1, x_2, x_3, x_4 be the corresponding displacements, then to form $(X_1X_2X_3X_4)$ we have from (6)

$$\left\{\begin{aligned}
x_3-x_1&=\frac{(AD-BC)(y_3-y_1)}{(Cy_3+D)(Cy_1+D)},\\
x_3-x_2&=\frac{(AD-BC)(y_3-y_2)}{(Cy_3+D)(Cy_2+D)},\\
x_4-x_1&=\frac{(AD-BC)(y_4-y_1)}{(Cy_4+D)(Cy_1+D)},\\
x_4-x_2&=\frac{(AD-BC)(y_4-y_2)}{(Cy_4+D)(Cy_2+D)},
\end{aligned}\right. \tag{7}$$

and by division

$$\frac{x_3-x_1}{x_3-x_2}\bigg/\frac{x_4-x_1}{x_4-x_2}=\frac{y_3-y_1}{y_3-y_2}\bigg/\frac{y_4-y_1}{y_4-y_2},$$

or

$$(X_1X_2X_3X_4)=(Y_1Y_2Y_3Y_4),$$

which is a property of a projective transformation. To prove that (6) is of the form (5), we find the double-points of (6) by putting $y=x$; then (6) becomes

$$Cx^2-(A-D)x-B=0; \tag{8}$$

hence, designating the roots of this equation by a and b,

$$\left\{\begin{aligned}
a&=\frac{A-D+\sqrt{(A-D)^2+4BC}}{2C},\\
b&=\frac{A-D-\sqrt{(A-D)^2+4BC}}{2C}.
\end{aligned}\right. \tag{9}$$

The transformation (6) has therefore two double-points. Putting in the first and third equations of (7) $x_3=y_3=a$ and $x_4=y_4=b$, it is found by division that

$$(10) \qquad \frac{a-x_1}{a-y_1}\Big/\frac{b-x_1}{b-y_1}=\frac{Cb+D}{Ca+D}=k \text{ (constant).}$$

Thus we find that any pair of corresponding points of the transformation (6) forms a constant anharmonic ratio with its double-points; such a transformation is projective. In deriving equation (5) it was assumed that A and B are real points. Assuming a projective transformation of the form (6), where A, B, C, D are real coefficients, it may happen that the double-points given by (9) are imaginary. In fact there are three possibilities for the double points. According as

$$(11) \qquad (A-D)^2+4BC \gtreqless 0,$$

a and b, or the double-points, are *real*, *real and coincident*, or *imaginary*, and the transformations are then called *hyperbolic*, *parabolic*, or *elliptic*.

4. We shall next show that *two projective ranges are determined by three pairs of corresponding points* X_1, Y_1; X_2, Y_2; X_3, Y_3, whose positions are determined by the coordinates x_1, y_1; x_2, y_2, . . . If these points are corresponding in two projective ranges, their coordinates must satisfy some relation of the form

$$x=\frac{ay+b}{cy+d}$$

or

$$cxy+dx-ay-b=0.$$

To determine the ratios $\frac{c}{b}, \frac{d}{b}, \frac{a}{b}$, which evidently determine the transformation, we have the conditions

$$\frac{c}{b}x_1y_1+\frac{d}{b}x_1-\frac{a}{b}y_1-1=0,$$

$$\frac{c}{b}x_2y_2+\frac{d}{b}x_2-\frac{a}{b}y_2-1=0,$$

$$\frac{c}{b}x_3y_3+\frac{d}{b}x_3-\frac{a}{b}y_3-1=0.$$

These are three equations with the three required ratios as unknown quantities. These are therefore uniformly determined by the x's and y's and are in determinant form:

$$\frac{c}{b}=\frac{\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}}{\begin{vmatrix}x_1y_1 & x_1 & y_1\\ x_2y_2 & x_2 & y_2\\ x_3y_3 & x_3 & y_3\end{vmatrix}},\quad \frac{d}{b}=\frac{\begin{vmatrix}x_1y_1 & 1 & y_1\\ x_2y_2 & 1 & y_2\\ x_3y_3 & 1 & y_3\end{vmatrix}}{\begin{vmatrix}x_1y_1 & x_1 & y_1\\ x_2y_2 & x_2 & y_2\\ x_3y_3 & x_3 & y_3\end{vmatrix}},\quad \frac{a}{b}=\frac{\begin{vmatrix}x_1y_1 & -x_1 & 1\\ x_2y_2 & -x_2 & 1\\ x_3y_3 & -x_3 & 1\end{vmatrix}}{\begin{vmatrix}x_1y_1 & x_1 & y_1\\ x_2y_2 & x_2 & y_2\\ x_3y_3 & x_3 & y_3\end{vmatrix}}.$$

This proposition is also geometrically clear. In two projective ranges any four points of one range have the same anharmonic ratio as the four corresponding points of the other range. Hence, choosing any fourth point x_4, then there is clearly only one point Y_4, so that

$$(X_1X_2X_3X_4)=(Y_1Y_2Y_3Y_4);$$

i.e., three pairs of points determine the projectivity.

As an exercise assume the case of two coincident projective ranges for which the infinitely distant point is self-corresponding. Let $x_1=y_1$ determine this infinitely distant point. From the above expressions we find $\frac{c}{b}=0$, while $\frac{d}{b}$ and $\frac{a}{b}$ are finite. The projective transformation assumes the form

$$x=\alpha y+\beta;$$

i.e., what is called a linear transformation.

5. It is beyond the limit of this book to discuss all special cases of projective transformations of a straight line in detail. We shall indicate one of its properties which is of extreme importance in modern geometry, and then discuss the special case of involution. Let a point x be transformed into a point x' by the projective transformation

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}. \tag{12}$$

Transform x' into a point x'' by another transformation of the same kind:

$$x'' = \frac{\alpha_1 x' + \beta_1}{\gamma_1 x' + \delta_1}. \tag{13}$$

The result of these two operations is

$$x'' = \frac{(\alpha\alpha_1 + \gamma\beta_1)x + (\beta\alpha_1 + \delta\beta_1)}{(\alpha\gamma_1 + \gamma\delta_1)x + (\beta\gamma_1 + \delta\delta_1)}, \tag{14}$$

which shows that x'' is obtained from x by a projective transformation of the form (12). Hence one, two, or more operations of the form (12) in succession are equivalent to an operation of the same kind. Giving α, β, γ, δ all possible real values, (12) depends upon the *three* ratios $\frac{\alpha}{\delta}$, $\frac{\beta}{\delta}$, $\frac{\gamma}{\delta}$, so that (12) represents a triply infinite number of projective transformations. For this reason it is said that *all projective transformations of a straight line form a continuous three-termed group* (*dreigliedrig*).[1]

§ 3. Involution.[2]

In case that the constant anharmonic ratio k in equation (4) of the foregoing paragraph is -1,

---

[1] Sophus Lie: Vorlesungen über continuierliche Gruppen.

[2] First systematically studied by Desargues (Brouillon projet, etc.).

(1) $$\frac{x-a}{x-b}=-\frac{y-a}{y-b}, \quad \text{or}$$

(2) $$x=\frac{(a+b)y-2ab}{2y-(a+b)}.$$

In these equations x and y can be interchanged without affecting (1) or (2). The ratio $(ABXY)=-1$ is called a *harmonic* ratio and (1) and (2) represent an *involutoric transformation.* To the point at infinity, $y=\infty$, corresponds the point $x=\frac{a+b}{2}$; i.e., the point bisecting the distance AB between the double-points. It is called the *middle point* of the involution. Designating this point by M, it is found that

(3) $$MX\cdot MY=\frac{(a-b)^2}{4}=\frac{\overline{AB}^2}{4};$$

i.e., *the product of the displacements of two corresponding points of an involution from the middle point is constant and equals the square of the displacement of either double-point from the middle point.*

Equation (2) may always be written in the form

(4) $$x=\frac{\alpha y-\beta}{\gamma y-\alpha},$$

if a and b are given by the values

$$a=\frac{\alpha}{\gamma}+\sqrt{\frac{\alpha^2}{\gamma^2}-\frac{\beta}{\gamma}}, \quad b=\frac{\alpha}{\gamma}-\sqrt{\frac{\alpha^2}{\gamma^2}-\frac{\beta}{\gamma}}.$$

As these expressions define the double-points, they must also result directly from (4). For the double-points $x=y$; hence from (4)

$$\gamma x^2-2\alpha x+\beta=0.$$

The roots of this equation are indeed identical with the previous values of a and b. If $\frac{\beta}{\gamma} > \frac{\alpha^2}{\gamma^2}$, a and b are conjugate complex numbers, i.e., the double-points of the involution are imaginary. In this case the middle point M of the involution is still real, since $\frac{a+b}{2} = \frac{\alpha}{\gamma}$, and

$$MX \cdot MY = \frac{(a-b)^2}{4} = \frac{\alpha^2}{\gamma^2} - \frac{\beta}{\gamma} < 0;$$

X and Y are on different sides of M. For $\frac{\beta}{\gamma} = \frac{\alpha^2}{\gamma^2}$ the double-points coincide, and $MX \cdot MY = 0$; every point corresponds to M. According to these results involution has been classified as hyper-

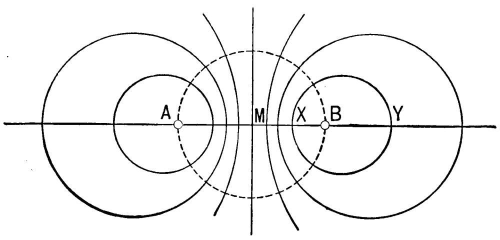


FIG. 4.

bolic in case of real double-points, elliptic in case of imaginary double-points, parabolic in case of coinciding double-points.

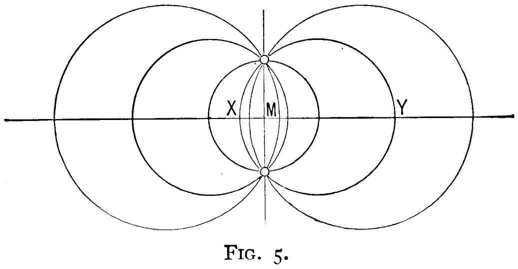


FIG. 5.

Geometrically, the different cases may be obtained as intersections of a straight line with coaxial systems of circles. Figs. 4,

5, and 6 represent hyperbolic, elliptic, and parabolic involutions respectively. In the first the points of a pair, XY, are always on the same side of M, and move in opposite directions; in the second they are on different sides of M, and one of the points (X) is

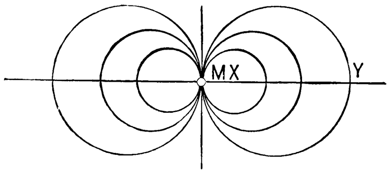


FIG. 6.

within the distance AB and the other without. Corresponding points move in the same direction. We have seen that an involution on a straight line is determined by the transformation

$$(5) \qquad x' = \frac{ax+b}{cx-a}, \quad \text{or}$$

$$(6) \qquad cxx' - a(x+x') - b = 0,$$

which shows that *an involution is determined by two pairs*, since there are only two essential constants in (6).

Suppose that in a projective transformation

$$x' = \frac{ax+b}{cx+d}, \quad \text{or}$$

$$cxx' + dx' - ax - b = 0,$$

the points x_1', x_1 may be interchanged without affecting the projectivity. The condition for this is

$$(7) \qquad cx_1x_1' + dx_1' - ax_1 - b = 0,$$

$$(8) \qquad cx_1x_1' + dx_1 - ax_1' - b = 0.$$

By subtraction

(9) $$d(x_1 - x_1') + a(x_1 - x_1') = 0,$$

which can only be satisfied when $d = -a$, since $x_1 \neq x_1'$. The condition $d = -a$, however, implies involution, hence the theorem:

If a projective transformation contains a pair whose values may be interchanged without altering the transformation, it is an involution. Thus if $x_1 x_1'$ be a pair,

(10) $$(ABX_1X_1') = ABX_1'X_1) = -1.$$

§ 4. Projective Pencils of Rays.

Let a, b, c, d be four rays (straight lines) passing through a common point, and (ab), (bc), etc., the angles included by the rays a and b, b and c, etc., so that also here $(ab) = -(ba)$; $(ab) + (bc) + (ca) = 0$.

In analogy with the anharmonic ratio of four points, the anharmonic ratio of these rays is

(1) $$\frac{\sin (ac)}{\sin (bc)} \Big/ \frac{\sin (ad)}{\sin (bd)} = k,$$

and may be designated by $(abcd) = k$.

What has been said about the permutations of four points applies without alteration to four rays. Consider now four concurrent rays a, b, c, d passing through four points A, B, C, D of a straight line, respectively. From Fig. 7 it is seen that

$$\frac{\sin (ac)}{\sin (bc)} \Big/ \frac{\sin (ad)}{\sin (bd)} = \frac{CN}{CP} \Big/ \frac{DM}{DO},$$

DM and CN being $\perp$ a and DO and CP $\perp$ to b. But

$$\frac{CN}{DM} = \frac{AC}{AD} \quad \text{and} \quad \frac{CP}{DO} = \frac{BC}{BD}; \quad \text{hence}$$

$$\frac{CN}{CP} \Big/ \frac{DM}{DO} = \frac{AC}{BC} \Big/ \frac{AD}{BD} \quad \text{and}$$

(2) $$(abcd) = (ABCD).$$

This important result may be stated by the theorem:

The anharmonic ratio of any four concurrent rays is equal to the anharmonic ratio of four points formed by the intersection of any transversal with these rays. (Pappus.)

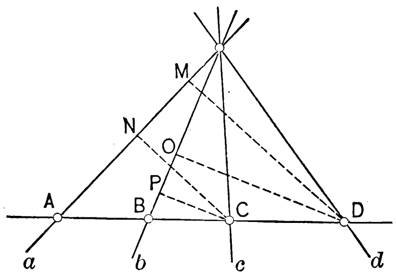


FIG. 7.

In other words, if the rays a, b, c, d are considered as projecting rays in a central projection, *such a projection does not change the anharmonic ratio of four points.*

A system of rays in a plane and passing through the same point is called a *pencil of rays.*[1] By the above theorem all properties of projective ranges of points may be transferred to projective pencils of rays.

In order to obtain an analytic expression for the rays of two projective pencils with the same vertices, assume the line representing a projective range of points as the X-axis and the origin of the range as the origin of a Cartesian system. Let V, with the coordinates m and n, be the center of a pencil, then the equations of the rays passing through the double points A and B of the transformation

$$(3) \qquad x=\frac{(a-bk)y-ab(1-k)}{(1-k)y-(b-ak)} \qquad \text{(eq. 5, § 2)}$$

[1] CREMONA, loc. cit., p. 22. In the translation of Reye's Geometrie der Lage the term "sheaf of rays" is used, while in Cremona's treatise "sheaf of rays" or "planes" means all rays or planes passing through a point in space. Ger. *Strahlenbüschel*. Fr. *Faisceaux*.

are

(4)
$$nx+(a-m)y-an=0,$$

(5)
$$nx+(b-m)y-bn=0.$$

Multiplying (5) by λ and subtracting from (4), the equation of a third ray through V results:

(6)
$$nx+\left(\frac{a-\lambda b}{1-\lambda}-m\right)y-\frac{a-\lambda b}{1-\lambda}n=0.$$

This ray intersects the X-axis in a point, say D, whose abscissa $d=\frac{a-\lambda b}{1-\lambda}$. To find the corresponding point C in transformation (3), put $y=d=\frac{a-\lambda b}{1-\lambda}$ in (3). This gives for the abscissa c of C the value $c=\frac{a-\lambda bk}{1-\lambda k}$, so that the equation of the ray passing through C becomes

(7)
$$nx+\left(\frac{a-\lambda bk}{1-\lambda k}-m\right)y-\frac{a-\lambda bk}{1-\lambda k}n=0.$$

Comparing equations (6) and (7) with those of (4) and (5), we find, if (4) and (5) are written $u=0$, $v=0$, that (6) and (7), the equations of the rays VD and VC, are

(8)
$$u-\lambda v=0,$$

(9)
$$u-\lambda kv=0.$$

For each value of λ these equations represent a corresponding pair in a projective transformation of rays which is characterized by the anharmonic ratio k. In other words, for a variable λ, (8) and (9) represent two projective pencils.

§ 5. Involutoric Pencils.

In the case of involution the anharmonic ratio is $k = -1$, so that equations (8) and (9) of the previous paragraph become

$$(1)\qquad u - \lambda v = 0,$$

$$(2)\qquad u + \lambda v = 0;$$

i.e., if u and v are any two rays, the rays $u - \lambda v = 0$ and $u + \lambda v = 0$ are harmonic with regard to u and v. For $\lambda = 0$ and $\lambda = \infty$ the double-rays $u = 0$ and $v = 0$ of the involution are obtained. (1) and (2) define an involution of rays when λ varies from $-\infty$ to $+\infty$. Suppose

$$u \equiv ax + by + c = 0,$$

$$v \equiv a_1 x + b_1 y + c_1 = 0$$

be the equations of the double-rays, so that (1) and (2) assume the form

$$(3)\qquad (a - \lambda a_1)x + (b - \lambda b_1)y + c - \lambda c_1 = 0,$$

$$(4)\qquad (a + \lambda a_1)x + (b + \lambda b_1)y + c + \lambda c_1 = 0.$$

The trigonometric tangents of the angles of inclination with $+X$, or the slopes of (3) and (4), are

$$(5)\qquad m = -\frac{a - \lambda a_1}{b - \lambda b_1},$$

$$(6)\qquad m_1 = -\frac{a + \lambda a_1}{b + \lambda b_1};$$

hence the tangent of the angle ϕ included by (1) and (2) or (3) and (4) is

$$(7)\qquad \tan\phi = \frac{m - m_1}{1 + m \cdot m_1} = \frac{2(a_1 b - a b_1)\lambda}{a^2 + b^2 - \lambda^2(a_1^2 + b_1^2)}.$$

This shows again that for $\lambda=0$ and $\lambda=\infty$, $\tan\phi=0$, or $\phi=0$ ($180°$). In these cases the rays (3) and (4) coincide and the double-rays of the involution are obtained. Supposing that $a_1b-ab_1\neq 0$, which generally will be the case, we may ask for those values of λ which will make $\tan\phi=\infty$, or $\phi=\frac{\pi}{2}$, a right angle. From (7) we find for this condition

$$a^2+b^2-\lambda^2(a_1{}^2+b_1{}^2)=0, \quad \text{or}$$

(8) $$\lambda=\pm\sqrt{\frac{a^2+b^2}{a_1{}^2+b_1{}^2}},$$

which is always a real quantity. Whether we take the + or − sign for λ in (8), we obtain the same couple of equations (3) and (4); hence the theorem:

An involution of rays always contains one, but only one, rectangular pair.

We shall now discuss the case where $\tan\phi=\infty$, or $\phi=90°$, for all values of λ. In order that this be the case, the quantities a, b, a_1, b_1 must satisfy the conditions $a^2+b^2=0$, $a_1{}^2+b_1{}^2=0$, $a_1b-ab_1\neq 0$, or $b=\pm ia$, $b_1=\mp ia_1$, so that the equations

$$ax+by+c=0, \quad a_1x+b_1y+c_1=0$$

of the double-rays assume the forms

$$x+iy+\frac{c}{a}=0, \quad x-iy+\frac{c_1}{a_1}=0,$$

and are imaginary. We can dispose of the constants $\frac{c}{a}$ and $\frac{c_1}{a_1}$ in such a manner that the double-rays will pass through the real point (α, β). Their equations then become

(9) $$\begin{cases} u\equiv x+iy-(\alpha+i\beta)=0, \\ v\equiv x-iy-(\alpha-i\beta)=0. \end{cases}$$

The involution with these double-rays has only rectangular pairs. The equations of such a pair are

$$(10)\qquad \begin{cases} u-\lambda v=0, \\ u+\lambda v=0. \end{cases}$$

For real values of λ the pairs are imaginary, since (10) may be written

$$(11)\qquad \begin{cases} y-\beta=i\dfrac{1-\lambda}{1+\lambda}(x-\alpha), \\ y-\beta=i\dfrac{1+\lambda}{1-\lambda}(x-\alpha). \end{cases}$$

Putting $i\dfrac{1-\lambda}{1+\lambda}=\mu$, a real quantity, $\lambda=\dfrac{1+i\mu}{1-i\mu}$. Thus, if in (10) we give λ all imaginary values contained in the formula $\lambda=\dfrac{1+i\mu}{1-i\mu}$, where μ is any real quantity, the corresponding pairs (11) in the involution will be real and rectangular. Now an involution of rays has generally only one rectangular pair and is determined by two pairs, hence the theorem:

An involution of rays having more than one rectangular pair has all its pairs rectangular.

The double-rays of this involution are imaginary and pass through the two infinite points, which, as will be seen later on, are called the *circular points at infinity*, § 12.

If an involution of rays shall contain the rays joining the vertex with the circular points, i.e., the two rays with the slopes $+i$ and $-i$ as a pair, then according to (3) and (4) we must have

$$\frac{a-\lambda a_1}{b-\lambda b_1}=+i, \qquad \frac{a+\lambda a_1}{b+\lambda b_1}=-i,$$

or

$$\lambda(b_1 i-a_1)+a-bi=0,$$

$$\lambda(b_1 i+a_1)+a+bi=0.$$

These two equations must exist for the particular value of λ which makes the slopes of (3) and (4) $+i$ and $-i$. This can only be true under the condition

$$(b_1 i - a_1)(a+bi) - (b_1 i + a_1)(a - bi) = 0,$$

or

$$aa_1 = -bb_1.$$

Hence the theorem:

If an involution of rays contains the rays with the slopes $+i$ and $-i$ as a pair, then the double-rays of this involution are perpendicular to each other.

Conversely, it can easily be proved that *if the double-rays of an involution are perpendicular, then this involution contains the rays with the slopes $+i$ and $-i$ as a pair.* The slopes of the rays of any pair in an involution, as defined by (3) and (4), are $-\frac{a-\lambda a_1}{b-\lambda b_1}$ and $-\frac{a+\lambda a_1}{b+\lambda b_1}$. Consequently the tangents of the angles which these rays make with one of the double-rays, for instance $u=0$ $\left(\text{slope} -\frac{a}{b}\right)$, are

$$\frac{-\frac{a}{b}+\frac{a-\lambda a_1}{b-\lambda b_1}}{1+\frac{a}{b}\,\frac{a-\lambda a_1}{b-\lambda b_1}} = \frac{\lambda(ab_1 - a_1 b)}{a^2+b^2-\lambda(aa_1+bb_1)}$$

and

$$\frac{\frac{a}{b}-\frac{a+\lambda a_1}{b+\lambda b_1}}{1+\frac{a}{b}\cdot\frac{a+\lambda a_1}{b+\lambda b_1}} = \frac{\lambda(ab_1 - a_1 b)}{a^2+b^2+\lambda(aa_1+bb_1)}.$$

In case of perpendicular double-rays $aa_1+bb_1=0$, and these two tangents become equal. Hence the theorem:

In case of perpendicular double-rays, the angles of all pairs of the involution are bisected by the double-rays. In such an involution two rays chosen from each of two pairs include the same angle as the remaining two rays of the two pairs.

§ 6. Product of Projective Pencils and Ranges.[1]

1. In § 4 it has been shown that the equations of two projective pencils of rays with the same vertex may always be written in the form

(1) $$u - \lambda v = 0,$$

(2) $$u - \lambda k v = 0.$$

For every value of λ these equations represent a corresponding pair of rays in a projective transformation which is characterized by the anharmonic ratio k. In other words, for a variable λ (1) and (2) represent two projective pencils. Now the second pencil (2) may be moved into any other part of the plane without ceasing to be projective with regard to (1). This operation does evidently not change the general form of (2); only the expressions u and v are transformed into new expressions r and s. These represent two rays intersecting each other in the vertex of the moved pencil. Thus the equations of the two pencils are

(3) $$u - \lambda v = 0,$$

(4) $$r - \lambda k s = 0.$$

For each value of λ there are two rays which intersect each other in a certain point P. If λ successively assumes all values between $-\infty$ and $+\infty$, P describes a locus whose equation is obtained by eliminating λ between (3) and (4). This gives

(5) $$vr - kus = 0,$$

[1] The conception of pencils of curves and surfaces represented by equations of the form $P + \lambda Q = 0$ is due to LAMÉ, who introduced it in his article, *Sur les intersections des lignes et des surfaces*, Gergonne's Annales, Vol. VII, 1816–17, pp. 229–240. The generation of conics by projective pencils and ranges is due to STEINER (Systematische Entwickelung, etc.), who called it his "steam-engine" (Dampfmaschine).

an equation of the second degree in x and y, since u, v, r, s are linear in x and y. Hence the theorem:

The product of two projective pencils of rays with separate vertices is a curve of the second order.

2. Each linear expression, like $ax+by+c=0$, depends on two independent coefficients, so that the equation $vr-kus=0$ contains eight independent coefficients. Arranging in (5) the terms according to powers of x and y, an equation of the form

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \tag{6}$$

is obtained, where a, b, c, d, e, f are expressed in terms of the coefficients of $u=0$, $v=0$, $r=0$, $s=0$.

As (6) contains only five independent coefficients, it is clear that the eight coefficients in $u=0$, $v=0$, $r=0$, $s=0$ may always be selected in such a manner that (5) becomes identical with any equation of the form (6). See problem 11 in § 7.

Hence the theorem:

Every curve of the second order may be considered as the product of two projective pencils of rays.

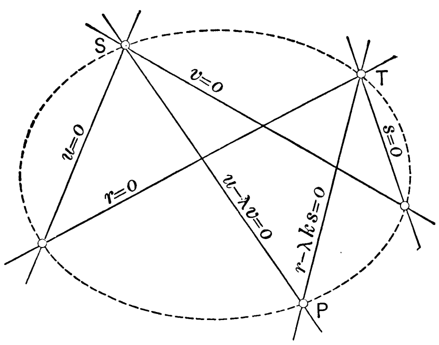


Fig. 8.

(5) is satisfied by $u=v=0$ and $r=s=0$, also by $u=r=0$ and by $v=s=0$; i.e., the curve of the second order passes through the vertices of the projective pencils and also through

the points of intersection of the rays v, s and u, r, Fig. 8. If, therefore, we want to write (6) in the form (5), we have to choose two points S and T on the given curve (6). Suppose that $u=0$ and $v=0$ contain S, and $r=0$ and $s=0$ T, then u, v, r, s, each only depend on one coefficient (slope), so that (5) depends on these four coefficients and k, which makes five coefficients in all. These coefficients may therefore be uniquely determined so that (5) represents or is identical with (6). Now two points S and T may be chosen in $\frac{\infty(-1)\infty}{2}=\infty^2$ different rays on a curve. Each of these determines a different but unique form of (5). The previous theorem may therefore be stated as follows:

Every curve of the second order may be produced in a doubly infinite number of ways by two projective pencils.

At the same time we have proved the theorem:

If two fixed points S and T be joined to a point P which describes a curve of the second order through S and T, the pencils (SP) and (TP) about S and T as vertices are projective.

3. As the general equation of a curve of the second degree depends upon five independent constants, it is clear that five points of the curve determine it. Designating the coordinates of one of these points by x_i, y_i, there is

$$ax_i^2+2bx_iy_i+cy_i^2+2dx_i+2ey_i+f=0,$$
$$i=1,\ 2,\ 3,\ 4,\ 5.$$

These are five equations with five unknown quantities $\frac{a}{f}$, $\frac{2b}{f}$, $\frac{c}{f}$, $\frac{2d}{f}$, $\frac{2e}{f}$, which may be found by the usual method. Hence the theorem:

A curve of the second order is determined by five points.

In a similar manner it may be proved that two projective ranges of points can be represented by the equations

$$\alpha-\mu\beta=0, \tag{7}$$

$$\gamma-\mu\kappa\delta=0, \tag{8}$$

where α, β, γ, δ are the line-equations [1] of four points in a plane. Assuming the knowledge of line-coordinates, the proof may be made without difficulty and may be left to the reader.

For every value of μ there are two points which determine a straight line. If μ successively assumes all real values, this line envelops a curve whose equation is obtained by eliminating μ from (7) and (8) and which is

$$\beta\gamma - \kappa\alpha\delta = 0. \tag{9}$$

This is an equation of the second degree in line-coordinates and consequently represents a curve of the second class.[2] In analogy with the previous statements we have also the theorems:

Every curve of the second class may be produced in a doubly infinite number of ways by two projective ranges.

If two fixed tangents S and T be intersected by a line P which envelops a curve of the second-class tangent to S and T, the ranges (SP) and (TP) on S and T as bases are projective.

A curve of the second class is determined by five tangents.

§ 7. Exercises and Problems.

1. Assuming $(ABCD)=k$, find the values of the other twenty-three ratios which may be formed with the four points $ABCD$.

2. Do the same when $(ABCD)=+1, -1$.

3. If X_1, X_2, X_3, X_4 and Y_1, Y_2, Y_3, Y_4 are corresponding points in a projective transformation, verify the relation

$$(X_1X_2X_3X_4)=(Y_1Y_2Y_3Y_4) \quad \text{by using}$$

$$y=\frac{ax+b}{cx+d}.$$

---

[1] Line-coordinates of a line are the negative reciprocal intercepts of this line with the coordinate axes. The reader is referred to SALMON-FIEDLER: *Analytische Geometrie der Kegelschnitte*, 6. ed., Vol. I, pp. 120–128.

[2] This statement stands for a definition.

4. If the double-points of an involution are $a=0$ and $b=\infty$, prove that the involutoric transformation has the form $x+y=0$.

5. If x and y are a pair in an involution with the double-points a and b, prove the relation

$$(x-a)(y-x)+(x-b)(y-x)+2(x-a)(x-b)=0.$$

6. Also establish the relation

$$\frac{1}{x-a}+\frac{1}{x-b}=\frac{2}{x-y}.$$

7. Prove that the middle point of an involution is always real.

8. What is the form of an involutoric transformation if the double-points are $+a$ and $-a$?

9. An involutoric transformation referred to its center as an origin may be represented by $x \cdot y=k^2$, where $\pm k$ locates the double-points. (k may be real or imaginary.) Prove that the anharmonic ratio of the points represented by $x,\ y, +k, -k$ is -1.

10. Prove that the rectangular pair of an involution of rays bisect the angles formed by the double-rays.

11. The equation of a circle

$$x^2+y^2=R^2$$

is given, and on it the points $(+r,\ 0)$ and $(-r,\ 0)$. Find the equivalent equation

$$vr-kus=0,$$

where the pencils $u-\lambda v=0$ and $r-\lambda ks=0$ have the given points as vertices.

§ 8. The Complete Quadrilateral.[1]

In § 5 it has been found that if p and q are linear expressions in x and y,

[1] See *Elemente der analytischen Geometrie* by F. JOACHIMSTHAL, pp. 131–142.

$$p=0,$$
$$q=0,$$
$$\left.\begin{array}{l}\alpha p+\beta q=0\\ \alpha p-\beta q=0\end{array}\right\}, \quad \left(\frac{\beta}{\alpha}=\lambda\right),$$

always are the equations of four harmonic rays of a pencil. By means of this theorem it is now possible to study the harmonic properties of the complete quadrilateral. Let $p=0$, $q=0$ and

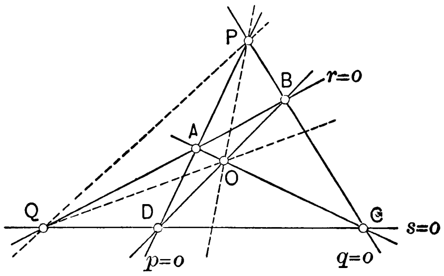


Fig. 9.

$r=0$, $s=0$ be the equations of two pairs of rays, Fig. 9. The equations of the rays passing through the vertices of these pairs are of the form $\alpha p+\beta q=0$ and $\alpha' r+\beta' s=0$ respectively. For certain values of α, β and α', β' these equations may both represent a ray passing through the vertices of both pairs, so that we have the identity

(1) $$\alpha p+\beta q=\alpha' r+\beta' s.$$

From this the further identities

(2) $$\alpha p-\beta' s=\alpha' r-\beta q,$$

(3) $$\beta' s-\beta q=\alpha p-\alpha' r$$

follow. Identity (2) represents a straight line through the points of intersection of the rays $p=0$ and $s=0$ and of the rays $r=0$ and $q=0$. The second represents a line passing through the points of intersection of $p=0$ and $r=0$ and of $s=0$ and $q=0$. Adding

(2) and (3) we get

(4) $$\alpha p-\beta q=\alpha p-\beta q,$$

i.e., the equation of a ray passing through O and P. The form of the equation shows that the ray is harmonic to the ray PQ with regard to the rays PC and PD.

Identity (3) may be written $\alpha p-\alpha' r=\beta' s-\beta q$. Subtracting this from (2), there results the new identity

(5) $$\alpha' r-\beta' s=\alpha' r-\beta' s.$$

This is the equation of a ray through O and Q. The form of the equation shows that the ray QO is harmonic to the ray QP with respect to the rays QB and QC. As (4) and (5) result from (2) and (3) by addition and subtraction, it is proved that OP, OQ and AC, BD are harmonic pairs. PC, PD; QB, QC; BD, AC are called the sides, and OP, OQ, QP the diagonals, of the complete quadrilateral. The previous results may be summed up in the theorem:

In every complete quadrilateral a pair of sides always forms a harmonic pencil with the two concurrent diagonals.

From this it follows that two vertices, for instance C and D, are harmonically divided by the two diagonals PO and PQ.

Ex. 1. If $p=0$, $q=0$, $r=0$ are the equations of the sides of a triangle, prove that any line of its plane may be represented by an equation of the form $\alpha p+\beta q+\gamma r=0$.

Ex. 2. Let $p=0$, $q=0$, $r=0$ be the equations of the diagonals of the quadrilateral, prove that

$$\begin{aligned} \alpha p+\beta q+\gamma r&=0,\\ -\alpha p+\beta q+\gamma r&=0,\\ \alpha p-\beta q+\gamma r&=0,\\ \alpha p+\beta q-\gamma r&=0 \end{aligned}$$

are the equations of a quadrilateral having those diagonals.

§ 9. Perspective Pencils and Ranges.

In § 6 it has been found that the equations of two projective pencils of rays with the vertices S and T may be written in the form

$$(1) \qquad \begin{cases} u+\mu v=0, \\ r+\mu s=0,^{1} \end{cases}$$

where u and v are two rays through S, and r and s two rays through T. In general the product of these pencils is a curve of the second order with the equation

$$(2) \qquad us-rv=0.$$

Every value of μ gives two corresponding rays $u+\mu v=0$ and $r+\mu s=0$, which intersect each other in a certain point of the curve. We will now assume that the rays u, v through S and r, s through T are chosen in such a manner that there exists a value k of μ so that the two corresponding rays $u+kv=0$, $r+ks=0$ are identical, or that *the ray through the vertices S and T is self-corresponding*. In this case

$$(3) \qquad u+kv=r+ks.$$

Eliminating u between (3) and (2) gives

$$rs+ks^2-ksv-rv=0, \qquad \text{or}$$

$$(4) \qquad (r+ks)(s-v)=0.$$

Eliminating v between (3) and (2) gives

$$us-rs-\frac{1}{k}r^2+\frac{1}{k}ur=0, \qquad \text{or}$$

$$(5) \qquad \left(s+\frac{1}{k}r\right)(u-r)=0.$$

---

[1] Here $\mu\equiv-\lambda$ and $s\equiv ks$ in formulas (3) and (4), § 6.

Equations (3), (4), and (5) show that in this case the curve of the second order degenerates into two straight lines, one passing through S and T, the other passing through the points of intersection of $u=0$ and $r=0$ and of $v=0$ and $s=0$. Hence the theorem:

If the ray connecting the vertices of two projective pencils is self-corresponding, then the product of the two pencils consists of the self-corresponding ray and another straight line.

Two pencils of this kind are said to be in a perspective position, or simply in perspective.

Similar arguments in line-coordinates, which may be left as an exercise to the reader, lead without difficulty to the theorem:

If the point of intersection of two projective ranges is self-corresponding in both ranges, then the product (envelope) of these ranges consists of the self-corresponding point and another point.

Two ranges of this kind are said to be in a perspective position, or simply in perspective.

The line where corresponding rays of two perspective pencils meet is called *axis of perspective.* The point through which rays joining corresponding points of two projective ranges pass is called *center of perspective.*

Ex. Prove the proposition concerning perspective ranges of points analytically (line-coordinates) and geometrically.

§ 10. General Construction of Projective Pencils and Ranges.

In § 2, 4 it has been proved that a projective transformation is determined by three corresponding pairs. This applies to pencils as well as ranges. This fact and the results of the previous section make it possible to construct projective pencils and rays.

A. *Projective Pencils.*—Let a, b, c and a', b', c' be three pairs of corresponding rays through the vertices L and L' respectively, Fig. 10. These determine two projective pencils of rays through the points L and L'. Taking c and c' as bearers of two ranges of points, obtained by the intersections of a', b',

c', . . . and a, b, c with c and c', respectively, we have accordingly the projective ranges

$$(c \cdot a'b'c' \ldots) = (c' \cdot abc \ldots).$$

As the points (cc') and $(c'c)$ are identical, it follows that they are in perspective, i.e., the lines joining the points (ca') and $(c'a)$, (cb') and $(c'b)$, . . . are all concurrent, say at P.

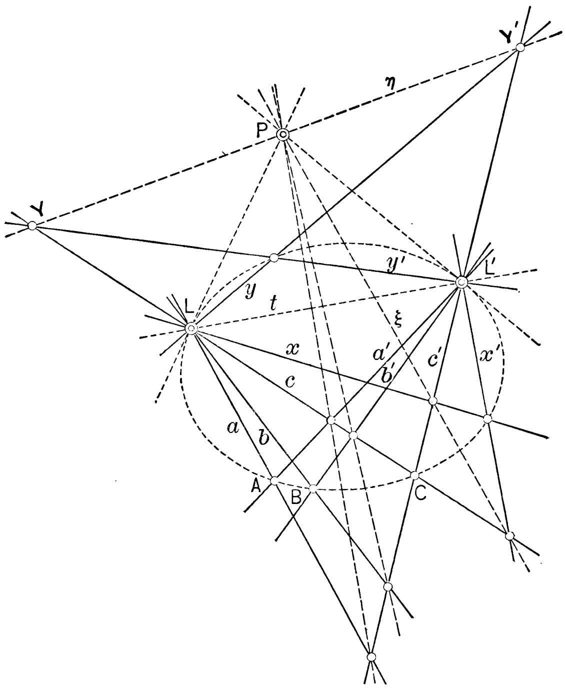


FIG. 10.

Hence, if any ray x of the first pencil is given, we know that the corresponding ray x' will be situated in such a manner that the line joining (xc') and $(x'c)$ will pass through P, and x' is found by joining the point of intersection of x and c' to P by a line ξ. The line joining L' to the point of intersection of ξ

and c is the required ray x'. In an entirely similar manner any ray of the second pencil may be assumed and the corresponding ray in the second pencil be constructed. Any ray η through P intersecting c and c' in two points Y and Y' gives rise to two corresponding rays LY and $L'Y'$, or y and y'. From this construction it is seen that *two projective pencils always admit of a third pencil which is in perspective with each of them.*

Now it is known that two projective pencils produce a curve of the second order in a unique manner. The six rays a, b, c; a', b', c' determine the five points L, L', (aa'), (bb'), (cc') of the curve, and every new pair of the construction like x and x', y and y', etc., determines a new point of the curve. The foregoing construction gives us therefore a means to construct any number of points of a curve of the second order, as soon as five of its points are given. If L, L', A, B, C—in any order—are the given five points, join L and L' each to A, B, C, thus obtaining the projective rays a, b, c and a', b', c'; then apply the construction and find as many points of the curve as desired.

The ray t joining L and L' is common to both pencils, but is not self-corresponding. Suppose t belongs to the pencil at L. To find its corresponding ray at L', produce t to its point of intersection T' with c'; join T' with P and find the point of intersection T of this line with c. The line joining L' with T is the required ray t'. Following this construction in Fig. 10 it is clear that $L'T$ is nothing else than PL'. Similarly, if t' is considered as belonging to the pencil at L', its corresponding ray will be PL. Taking a ray through either L or L', very close to t, and making the construction for the corresponding ray, supposing at the same time that the original ray passes to the limiting position of t, it is easily found that PL and $P'L'$ are the tangents from P to the curve of the second order.

B. *Projective Ranges.*—Let A, B, C and A', B', C' be three pairs of corresponding points on the lines l and l' respectively, Fig. 11. These determine two projective ranges of points on l and l'. Taking c and c' as vertices of pencils of rays, joining

A', B', C', . . . , and A, B, C, . . . , respectively, we obtain the projective pencils

$$(C \cdot A'B'C' \ldots) = (C' \cdot ABC \ldots)$$

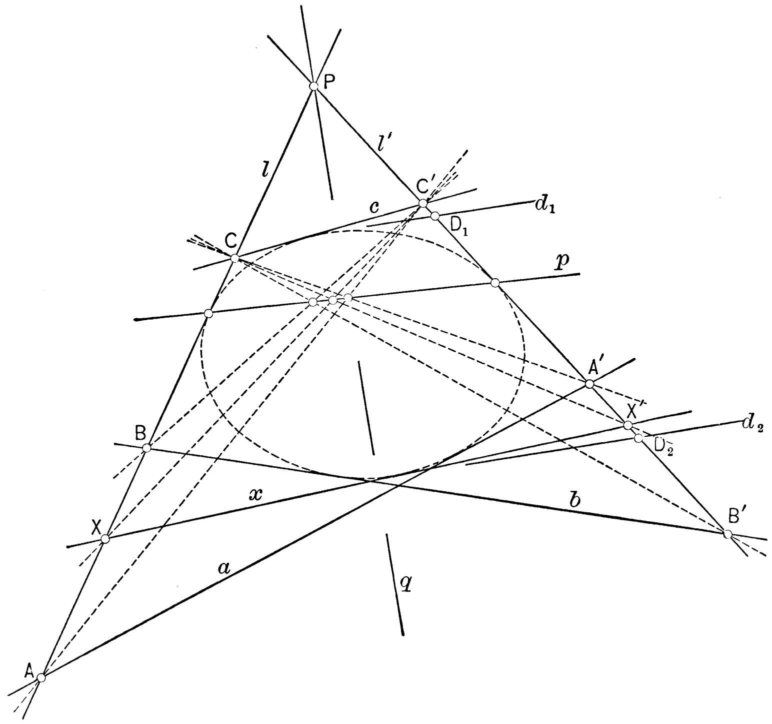


FIG. 11.

As the ray CC' or $C'C$ is common to both, it follows that they are in perspective; i.e., the points of intersection of the rays CA' and $C'A$, CB' and $C'B$, etc., are all on the same straight line, say p.

Hence, if any point X of the first range is given, the corresponding point X' is found by joining C' to X and finding the point of intersection of this joining-line with p. The line joining C to this latter point cuts l' in the required point X'. In an

entirely similar manner any point of the second range may be assumed and the corresponding point in the first be constructed. Any point in p gives rise to two corresponding points on l and l'. From this construction it is seen that *two projective ranges always admit of a third range which is in perspective with each of them.*

The line p intersects l and l' each in a point whose corresponding points coincide with the point of intersection of l and l'. Again, l, l', AA', BB', and CC' are five tangents to a curve of the second class and the foregoing construction makes it possible—by joining X and X'—to construct any number of tangents. The line of perspective cuts l and l' in their points of tangency.

§ 11. Exercises and Problems.

1. Given five points of a curve of the second order; construct five other points, each being situated between two of the given points, i.e., one between A and B, one between B and C, etc.

2. Construct the tangents at each of the given points.

3. Given five tangents of a curve of the second class; construct any number of other tangents and the points of tangency of the given tangents.

4. Two projective ranges $(ABC\ldots)=(A'B'C'\ldots)$ on the lines l and l' determine a curve K of the second class having AA', BB', CC', ... as tangents. Conversely, every tangent x of K cuts l and l' in two corresponding points of the ranges. If we now turn l' about its point of intersection P with l through the space containing K, Fig. 11, two coincident projective ranges arise. To obtain the double points of these, § 2, draw the bisector q of the angle between l and l'. The two tangents d_1 and d_2, perpendicular to q, intersect either l or l' in the required double- or self-corresponding points D_1 and D_2.[1]

[1] This construction has been successfully used as a base for the synthetic treatment of the projective continuous groups by Professor NEWSON and myself. See Kansas University Quarterly, Vol. IV, p. 243 and Vol. V, No. 1.

5. What position must K have with respect to l and l' in order to make the projective ranges on l and l' involutoric?

6. Show that with K as a circle the projective ranges are involutoric.

7. Assume five points L, L', A, B, C of a curve of the second order in such a manner that the respective pencils are involutoric.

8. Verify problems 1 and 2 on a given circle.

9. Prove NEWTON'S theorem (Principia, lib. i., lemma xxi).

If two angles AOS and $AO'S$ of given magnitude turn about their respective vertices O and O' in such a way that the point of intersection S of one pair of arms lies always on a fixed straight line u, the point of intersection of the other pair of arms will describe a conic (Cremona's statement).

10. Prove MACLAURIN'S theorem (Phil. Trans. of the Royal Society of London for 1735).

If a triangle $C'PQ$ move in such a way that its sides PQ, QC', $C'P$ turn round three fixed points R, A, B, respectively, while two of its vertices P, Q slide along two fixed straight lines CB', CA', respectively, then the remaining vertex C' will describe a conic which passes through the following five points, viz., the two given points A and B, the point of intersection B' of the straight lines AR and CB', and the point of intersection A' of the straight lines BR and CA'.

§ 12. Projective Properties of the Circle.

To specialize the results concerning projective pencils for the circle it is simplest to depart from the equation of the circle

$$x^2+y^2-r^2=0. \tag{1}$$

This may be written in the form

$$(x+iy)(x-iy)-r^2=0,$$

which itself may be considered as the result of the elimination of λ between the projective pencils

$$\begin{cases} x+iy+\lambda r=0, \\ r+\lambda(x-iy)=0. \end{cases} \tag{2}$$

The vertices of these imaginary pencils are the points of intersection of the line at infinity, $r=0$, with the rays $x+iy=0$ and $x-iy=0$. These points are called the circular *points at infinity.*[1] Taking the center of the circle at (a, b), the equation of the circle becomes

$$(x-a)^2+(y-b)^2-r^2=0, \quad \text{or}$$

$$\{(x-a)+i(y-b)\}\{(x-a)-i(y-b)\}-r^2=0. \tag{3}$$

Eq. (3) is the result of the elimination between the projective pencils

$$\begin{cases} x+iy-(a+ib)+\lambda r=0, \\ r+\lambda(x-iy-a+ib)=0, \end{cases} \tag{4}$$

and shows that all circles of the plane pass through the same circular points. As a curve of the second order is determined by five points, a circle must be determined by three points, two fixed points (the circular points) being given in advance.

A circle can also be produced by two projective pencils with real vertices. Graphically this proposition is evident. If, in Fig. 12, ST be a chord of a circle, all angles subtended by this chord are equal, i.e., $\angle ASC=\angle ATC$, $\angle BSC=\angle BTC$, etc. Hence

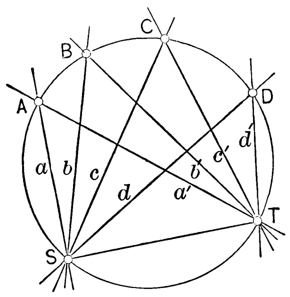


Fig. 12.

$$(abcd\ldots)=(a'b'c'd'\ldots);$$

the pencils at S and T are projective. Connecting any point in space with all points of the circle, a cone is obtained. Cutting this cone by any plane and passing planes through the vertex of this cone and the rays of the pencils through S and T, two new pencils of rays $(a_1b_1c_1d_1\ldots)$ and $(a_1'b_1'c_1'd_1'\ldots)$ are obtained on the intersecting plane, which are again projective. *Their product is*

[1] Introduced by Poncelet, loc. cit., p. 94.

therefore a curve of the second order; in this case a conic. It will be seen later on that *all curves of the second order are identical with all conics.*

To show how a circle may be described as an envelope, assume first the line-equation of a circle

$$u^2+v^2=\frac{1}{r^2}, \tag{5}$$

where u and v are the line-coordinates and r the radius. Equation (5) is the product of the two projective ranges

$$\begin{cases} u+iv+\dfrac{\lambda}{r}=0, \\ \dfrac{1}{r}+\lambda\,(u-iv)=0. \end{cases} \tag{6}$$

The coordinates of the line at infinity are $u=v=0$; consequently the points $u+iv=0$ and $u-iv=0$ are situated on the line at infinity. $\dfrac{1}{r}=0$ is the equation of the origin. Thus the projective ranges (6) are situated on the two imaginary straight lines joining the points $u+iv=0$ and $u-iv=0$ with the origin. These lines are tangent to the circle and they pass through the circular points at infinity. A translation does not change these results, so that the theorem may be stated:

The tangents to a circle from its center pass through the circular points at infinity.

In the case of real projective ranges producing a circle it is more convenient to assume the circle and to prove that it is the product of two projective ranges. Let in Fig. 13 $\angle OAA'=\alpha$, $\angle OBB'=\beta$, $\angle AOB=\phi$, and in a similar manner $\angle OA'A=\alpha'$, $\angle OB'B=\beta'$, $\angle A'OB'=\phi'$. There is $2\alpha+2\alpha'=\pi-\gamma$, $2\beta+2\beta'=\pi-\gamma$, hence $\beta-\alpha=\alpha'-\beta'$. But $\beta-\alpha=\phi$ and $\alpha'-\beta'=\phi'$, hence $\phi=\phi'$. This is true for any two tangents to the circle, so that the pencils $(O\cdot ABCD\ldots)$ and $(O\cdot A'B'C'D'\ldots)$ are projective. From this follows that the ranges formed by the points of inter-

section of all tangents with two fixed tangents are equal. Conversely, the product of these particular ranges is a circle, as might also be proved directly. The tangents and ranges of this example may again be connected to a point in space. Cutting this configuration by any plane in space, two projective ranges pro-

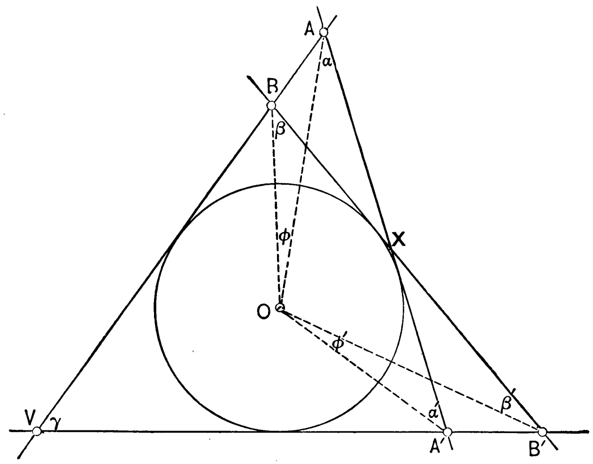


FIG. 13.

ducing a curve of the second class are obtained. It will be seen later on that *all curves of the second class are identical with all conics or curves of the second order.*

§ 13. Polar Involution of the Circle.

Through a given point A, Fig. 14, draw any ray intersecting a given circle in two points C and D. On this ray determine a point B in such a manner that the anharmonic ratio

$$(ABCD) = -1,$$

i.e., harmonic. If this operation is repeated for every ray passing through A, the points B on all these rays will form a certain locus which is a straight line, and which is called the *polar of*

the point A with regard to the given circle. The point A is called the *pole*. To prove this assume

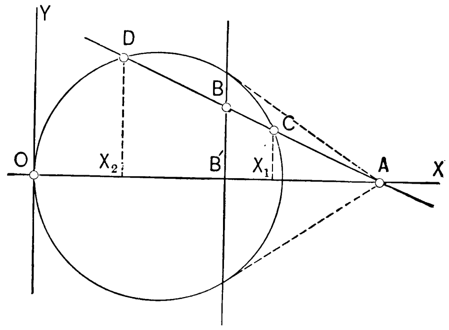


FIG. 14.

$$(1) \qquad x^2 - 2rx + y^2 = 0$$

as the equation of the circle and $(a, 0)$ as the coordinates of the point A. The special position of point and circle has no influence upon the generality of the result. The equation of any ray through A may be written

$$(2) \qquad y = (a - x)m.$$

Solving (1) and (2) it is found that the abscissæ x_1 and x_2 of the points of intersection C and D of the ray with the circle are

$$(3) \qquad \begin{cases} x_1 = \dfrac{r + am^2 + \sqrt{r^2 + 2arm^2 - a^2m^2}}{1 + m^2}, \\[2ex] x_2 = \dfrac{r + am^2 - \sqrt{r^2 + 2arm^2 - a^2m^2}}{1 + m^2}. \end{cases}$$

Now $$(AB'X_1X_2) = (ABCD) = -1,$$

or $$\frac{a - x_1}{b - x_1} = -\frac{a - x_2}{b - x_2},$$

from which $$b = \frac{a(x_1 + x_2) - 2x_1x_2}{2a - (x_1 + x_2)}.$$

Substituting the values for x_1 and x_2 in this expression, then

$$b=\frac{ar}{a-r};$$

i.e., the abscissa of B is independent of m and is therefore a constant. *The locus of the point B is consequently a straight line* parallel to the y-axis, *or perpendicular to the line joining the point A with the center of the circle.*

If A is without the circle, there are rays which do not cut the circle, or for which the points of intersection are imaginary. This is the case when $r^2+2arm^2-a^2m^2<0$, or $a^2m^2-2arm^2>r^2$, or $|m|>\left|\frac{r}{\sqrt{a^2-2ar}}\right|$; x_1 and x_2 are conjugate-imaginary, so that x_1+x_2 and x_1x_2 are real quantities and consequently also b is a real quantity. Hence if C and D are imaginary B is still real, and $(ABCD)=-1$. If $m=\frac{\pm r}{\sqrt{a^2-2ar}}$, the points C and D coincide and the rays through A become tangent to the circle, which are real when A is outside $(2r<a)$, and imaginary when A is inside $(2r>a)$. Hence the theorem:

The polar of a point with regard to a circle passes through the points of tangency (real or imaginary) from this point to the circle.

In the case of a pole within the circle the equation of the polar becomes

$$b=\frac{a}{\frac{a}{r}-1}.$$

The greatest value for a is in this case $2r$, so that up to this limit $\frac{a}{r}-1<1$ and $b>a$. The smallest value of b is for $a=2r$, i.e., $b=2r$. For $a<2r$, we have therefore always $b>2r$; the polar does not intersect the circle. For $a=2r$, $b=2r$, the pole coincides with the polar, which in this case becomes a tangent; i.e., *a tangent is the polar of its point of tangency* and *a point of tangency is the pole of the corresponding tangent.* For the center of the circle $a=r$ and $b=\infty$, the polar is the line at infinity. For

the tangents from the center $m = \pm i(a=r)$, so that the equations according to (2) become $x+iy=r$, $x-iy=r$. This shows again that *the tangents from the center of a circle touch the circle at its circular points,* a result obtained in the previous section.

§ 14. Continuation of § 13.

Taking any point, for instance B, on the polar of A, it is clear that the polar of B must pass through A, since A is harmonic to B with regard to C and D as the other pair. Thus the theorem:

The polar of a point which is situated on the polar of another point passes through the latter point. Conversely, the pole of a straight line which passes through the pole of a second line is situated on the latter.

From this it follows that the tangents at C and D intersect each other in a point of the polar of A. This point is evidently

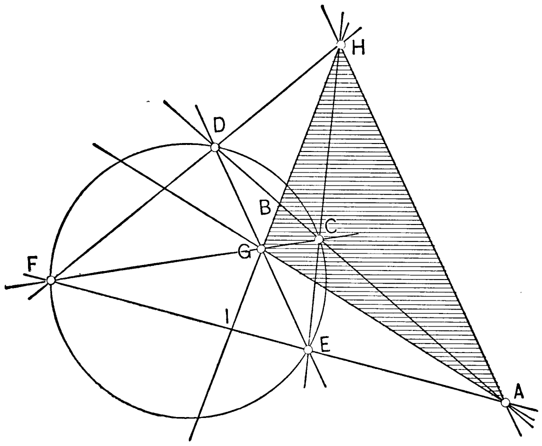


FIG. 15.

the pole of the ray ($ABCD$), through A. Using the results of § 8, concerning the complete quadrilateral, it is now easy to give a simple construction of the polar of a point, or of the pole of a straight line. Through A draw any two rays intersecting the

circle in the points C, D and E, F, Fig. 15. Connect C with F, and D with E, and find the point of intersection G of these connecting lines. In the same manner find the point of intersection H of the lines connecting C with E, and D with F. The line through G and H is the required polar of A. The proof is immediate, for $(ABCD) = (AEIF) = -1$, which is the condition that GH be the polar of A. The polar of H must pass through A, and since $(HGBI) = -1$, it follows that it also passes through G. Hence the polar of H is AG. The polar of G passes through A and H, hence AH is the polar of G. The triangle AGH possesses the important property that *the polar of each of its*

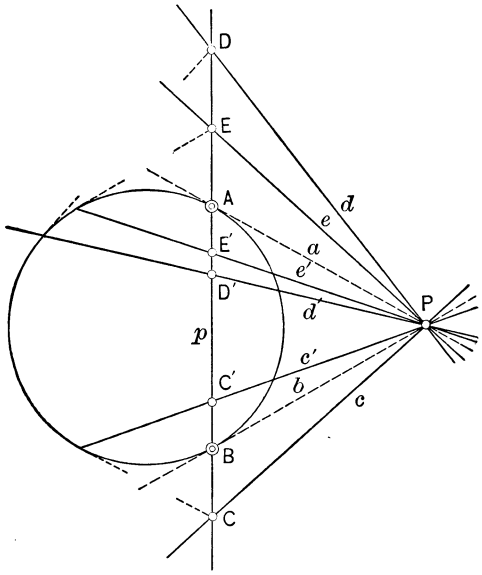


Fig. 16.

vertices is the opposite side in the triangle, and the pole of each side is the opposite vertex of this side. This triangle is called a *self-polar triangle with regard to the circle.*

Consider now in Fig. 16 the pole P and its polar p intersecting the circle in two points A and B. Through P draw any ray c intersecting p in C, and determine the pole C' of the

ray c. Then $(ABCC') = -1$. Designating the tangents from P to the circle by a and b, and the ray PC, which is the polar of C', by c', there is also $(abcc') = -1$. For every ray through P a pair of poles and a pair of polars are obtained which are harmonic to A and B, and to a and b, respectively. In this manner an *involution of coincident poles and polars* arises. In the case of the figure A and B are the real double-points, a and b the real double-rays of the involution. It is noticed that in this hyperbolic involution each pair is separated by the double-elements. Two pairs either exclude each other entirely, like CC' and DD', or include each other entirely, like DD' and EE'. If P were within the circle, we should have an elliptic involution, where two pairs always overlap each other. As an interesting example of this kind, consider the right-angle involution of the circle, Fig. 17.

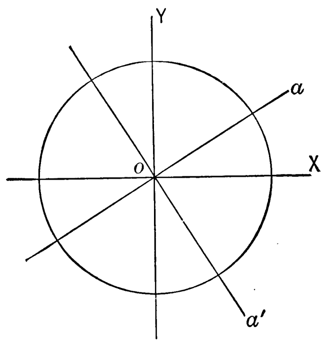


FIG. 17.

The polar of the center is the line at infinity. To every diameter a as a polar corresponds a pole A which is the infinite point of the perpendicular diameter a'. Thus a and a' are a pair of the polar involution about the center. In fact the rays of each pair are perpendicular to each other. To find the double-rays let

$$y = mx,$$

$$y = -\frac{1}{m}x,$$

be the equations of any pair. For a double-ray these equations must be identical. This is only possible when $m=-\frac{1}{m}$, or $m^2=-1$, which gives as the only possibilities $m_1=i$, $m_2=-i$. The equations of the double-rays are therefore $x+iy=0$ and $x-iy=0$. As they are the double-rays of a right-angle involution, the paradoxical result is obtained that *each of these rays is perpendicular to itself.* Geometrically this has no meaning.

Ex. 1. Construct a self-polar triangle having two poles within the circle.

Ex. 2. Discuss the elliptic pole and polar involution and make the necessary constructions.

Ex. 3. Explain the involutoric relation between an inscribed quadrilateral *ABCD* of a circle and the quadrilateral circumscribed at *A*, *B*, *C*, *D*.

CHAPTER II.

COLLINEATION.

§ 15. Central Projection.[1]

A central projection, or a perspective, is determined by the plane of projection (plane of the picture) and the center (eye). Assuming the plane of the paper as the plane of projection and any point in space as the center, it is possible to construct the perspective of any figure in space on this plane.

The center can most easily be located by a circle in the plane of projection. The radius of this circle is the distance of the center from the plane, and the center of the circle is the orthographic projection of the center upon the plane of projection. This circle has been introduced into geometry by Professor FIEDLER of Zürich, who calls it *distance-circle* [2] (Distanzkreis). In this section only the projections of figures in a plane will be considered and the geometrical laws involved in this

[1] *Historic Note.*—DESARGUES, whom Poncelet called the MONGE of his century, was the first to investigate the relation of central projection to the geometry of position; i.e., the purely projective properties of central projection (perspective), in his *Méthode universelle de mettre en perspective les objets donnés réellement* (Paris, 1636). These principles are also contained in the *Œuvres de Desargues réunies et analysées par Poudra*, Paris, 1864, Vol. I.

BROOK TAYLOR'S *New Principles of Linear Perspective*, London, 1715 and 1719, and J. H. LAMBERT'S work, *Die freie Perspektive, oder Anweisung jeden perspektivischen Aufriss von freien Stücken und ohne Grundriss zu verfertigen*, Zürich, 1759, II. part, 1774, contain also the fundamental principles of perspective.

For further information see the introductory chapter of WIENER'S *Darstellende Geometrie*, Leipzig, 1884–87, which contains a history of this science and a chapter on perspective in Vol. II; also FIEDLER'S *Darstellende Geometrie*, Vol. I.

[2] *D. Geometrie*, Vol. I, 1883, and *Cyclographie*, Chapter VIII. The method followed here is that of Fiedler.

projection explained. The plane of projection will be designated by π', and the arbitrary plane, whose perspective will be made, by π, Fig. 18. Let s be the line of intersection of π and π'. To obtain the projection P' of any point P in π, connect P with the center C and determine the point of intersection P' of this connecting line with π'. In a similar manner, the projection l' of a line l (RS) in π is obtained as the line of intersection of the plane, passing through C and l, with π'. From this construction the following fundamental laws are immediately clear:

To every point of π corresponds a point of π', and conversely, and both points lie on a ray through C.

To every straight line of π corresponds a straight line of π', and conversely, and both lines meet in a point of s.

To the line at infinity of π corresponds a line q' of π' which is parallel to s. Conversely, to the line r' at infinity of π' corresponds a line r parallel to s.

The plane π is usually determined by its trace s in π' and either of the lines r and q'. If a straight line l in π is given, intersecting s in S, the corresponding line l' is determined by drawing a line through C parallel to l and marking its point of intersection Q' with q'. It is apparent that Q' is the projection of the infinite point of l, and the projection of l consequently passes through S and Q'. Another way is to produce l to its point of intersection R with r and to join C with R. The line through S parallel to CR is l'. From the figure it is seen that $CRSQ'$ is a parallelogram and that

$$PS : PR = P'S : CR.$$

The planes through C parallel to π and π' form a space of a parallelepiped. Keeping π' fixed, it is possible to turn the planes π and the planes through C parallel to π and π' down into π', without changing s and q' and the distances of C and r, C and q', S and r, and S and q' in these planes.

After the motion there is still $CR \parallel$ and $= Q'S$, and $SP' = SP'$; consequently the distances PR and PS are not changed by the

motion. From this it follows that after the motion P' and the revolved position of P lie on a ray through the revolved position of C. The laws expressing the geometrical relation between the revolved and the projected figure are therefore the same as those

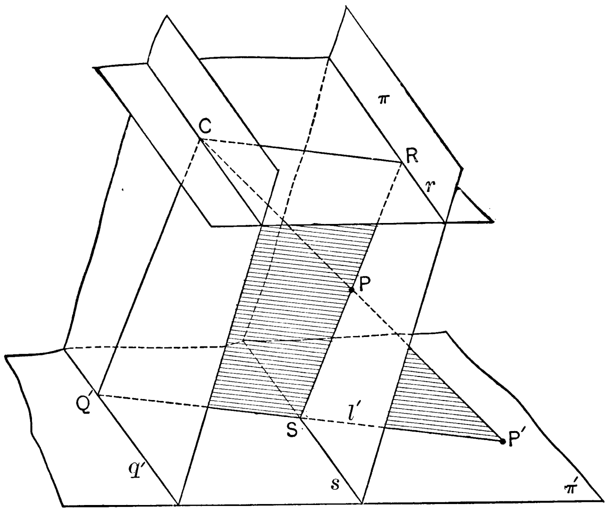


FIG. 18.

between the figure in space and its projection. After the rabattement, Fig. 18 assumes the form of Fig. 19.

In this figure l and l' are the two corresponding lines which with s and SC form a pencil of four rays through S. As this pencil is intersected by the rays CP and CQ, we have

$$(CLP'P)=(CMQ'Q).$$

The value of $(CMQ'Q)$ is $\dfrac{CQ'}{MQ'}=\dfrac{CO}{NO}=k$, say; i.e., entirely independent of the position of l, l', and CP. Thus, drawing any ray through C, intersecting s in S, and constructing any two corresponding points P and P' (rotated position of a point in π and its projection on π'), we have

$$(CSPP')=\text{const.}$$

Keeping CS fixed and constructing all possible pairs (P, P'), two coincident projective series of points are obtained having C and S as double-points. The different cases of central projection may be classified according to the position of the center and to the value of the constant k of the projection. Before entering upon

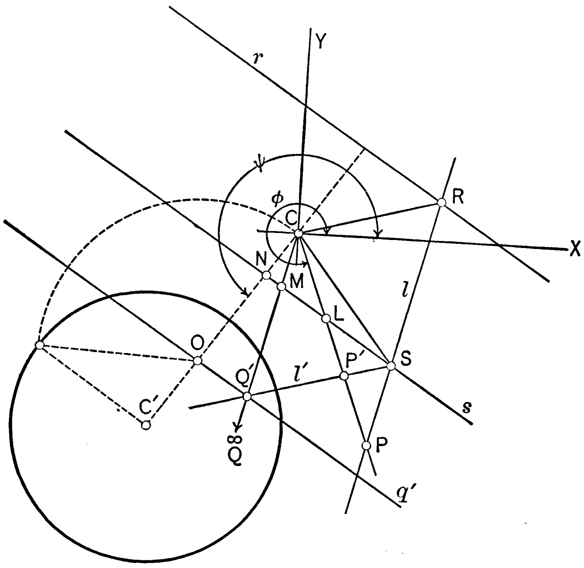


FIG. 19.

these details it is important to establish the analytical relation between a pair P, P'.

Ex. 1. In both figures 18 and 19 it is noticed that the distance between r and s is equal to the distance between C and q'. Assume the elements of a perspective as in Fig. 19, and draw the perspective of a triangle (a) which does not cut r; also of a triangle (b) which cuts r.

Ex. 2. Draw in a similar manner the perspective of a regular hexagon which is not cut by r.

Ex. 3. Draw the perspective of a circle

(a) which does not cut r; also of a circle

(b) which cuts r in two points; and of a circle

(c) which touches r.

Ex. 4. Draw the perspective of a system of concentric circles.

Ex. 5. Find l' when l is parallel to s.

Ex. 6. Construct the perspective of a circle having its center at C, Fig. 19.

Note.—In all these exercises the given figures are, of course, in the revolved position of π; i.e., in π'.

§ 16. Analytical Representation of Central Projection.

In Fig. 19 assume any two perpendicular lines through C as coordinate axes and designate the angle which the X-axis makes with CO by ψ, and its angle with CP by ϕ. Designate the coordinates of P and P' respectively by x, y and x', y'. Now

$$(CLPP')=k, \quad \text{or} \quad CP'=\frac{CP\cdot LP'}{k\cdot LP}, \quad \text{or} \quad CP'=\frac{CP(CP'-CL)}{k(CP-CL)}.$$

From this

$$CP'=\frac{CP\cdot CL}{k\cdot CL-(k-1)CP}.$$

Now

$$CP=\sqrt{x^2+y^2}, \quad CL=\frac{CN}{\cos(\psi-\phi)}, \text{ or, since}$$

$$\cos(\psi-\phi)=\cos\psi\cos\phi+\sin\psi\sin\phi=\frac{x\cos\psi}{\sqrt{x^2+y^2}}+\frac{y\sin\psi}{\sqrt{x^2+y^2}},$$

$$CL=\frac{CN\sqrt{x^2+y^2}}{x\cos\psi+y\sin\psi};$$

hence, by substitution in the above value for CP',

$$CP'=\frac{CN\cdot\sqrt{x^2+y^2}}{(1-k)\cos\psi\cdot x+(1-k)\sin\psi\cdot y+k\cdot CN}.$$

Now $x' = CP' \cdot \cos\phi$; $y' = CP' \cdot \sin\phi$; hence

$$\text{(I)} \qquad \begin{cases} x' = \dfrac{CN \cdot x}{(1-k)\cos\phi \cdot x + (1-k)\sin\phi \cdot y + k \cdot CN}, \\[2ex] y' = \dfrac{CN \cdot y}{(1-k)\cos\phi \cdot x + (1-k)\sin\phi \cdot y + k \cdot CN}. \end{cases}$$

In these expressions there are three arbitrary parameters: CN, k, ϕ. Conversely, if the transformation

$$\text{(II)} \qquad \begin{cases} x' = \dfrac{ax}{dx+ey+f}, \\[2ex] y' = \dfrac{ay}{dx+ey+f} \end{cases}$$

is given, it always represents a perspective. To prove this, it is sufficient to reduce (II) to the form (I). This can be done in one and only one way, by putting $\dfrac{f}{a} = k$,

$$\frac{(1-k)\cos\phi}{CN} = \frac{d}{a}, \quad \frac{(1-k)\sin\phi}{CN} = \frac{e}{a},$$

and as a consequence $\qquad \dfrac{e}{d} = \tan\phi.$

From this $CN = \dfrac{f-a}{\sqrt{e^2+d^2}}$. Equations (II) are the most general representation of a perspective. The points (x, y) in π for which (x', y') in π' become infinite are evidently situated in the line $dx+ey+f=0$. This is therefore the equation of the line r. For the line s we have $x=x'$, $y=y'$; hence from the first equation of (II)

$$dx^2+exy+fx=ax,$$

or

$$dx+ey+f-a=0,$$

as the equation of s.

Equations (II) may also be written in the form

$$x(dx'-a)+yex'+fx'=0,$$
$$xdy'+y(ey'-a)+fy'=0.$$

The condition that the values for x and y become infinite is

$$\begin{vmatrix} (dx'-a) & ex' \\ dy' & (ey'-a) \end{vmatrix} = 0,$$

or explicitly

$$dx'+ey'+a=0.$$

This is therefore the equation of q'.

In these calculations the coordinate-origin is C, so that $\frac{f-a}{\sqrt{d^2+e^2}}$ is the distance of s from C, or CN.

The distance of q' from C is $\frac{a}{\sqrt{d^2+e^2}}$, and that of r from C is $\frac{f}{\sqrt{d^2+e^2}}$. This naturally all agrees with Fig. 19 from which they were derived. It must be remarked that these formulas only hold when C is in finite regions.

§ 17. Special Cases of Central Projection.

A. Involution.—If in Fig. 18 the center C is situated in the bisecting plane of π and π' and if π is subsequently turned in the direction of the space between π and π' in which the bisector lies, then r will coincide with q', and after the rotation $CO=-NO$. C and q', r, in this case, are on the same side of s; $k=\frac{CO}{NO}=-1$. This perspective is called an involution, since in Fig. 19 to every line l of π corresponds a line l' of π'. If l' is considered as the rotated position of a line in π, then its corresponding line in π' when rotated will coincide with l. This fact immediately appears from the construction and also from equations (II), which in this case assume the form

$$\text{(III)} \qquad \begin{cases} x'=\dfrac{ax}{dx+ey-a}, \\[2ex] y'=\dfrac{ay}{dx+ey-a}, \end{cases}$$

or
$$dxx' - a(x+x') + ex'y = 0,$$
$$eyy' - a(y+y') + dxy' = 0.$$

Now in every perspective $x'y = xy'$, as is seen immediately by dividing both equations (II); hence these equations remain the same when x, y and x', y' are interchanged.

If the plane π' is turned in the opposite direction as in the involution, $k = +1$, and the revolved position of π' in this case is also obtained from that of an involution by a reflection on the s-axis.

B. Similitude.—This case arises when $\pi' \parallel \pi$; i.e., if s is infinitely distant; hence the equation of s must appear in the form $f - a = 0$ and $d = 0$, $e = 0$. Equations (II) now go over into

$$\text{(IV)} \qquad \begin{cases} x' = \dfrac{a}{f}x, \\ y' = \dfrac{a}{f}y. \end{cases}$$

When $k = \frac{f}{a}$ is positive, $(CLP'P) > 0$ and equals $\dfrac{CP'}{CP} = k$. P and P' are on the same side of C, and the center in space is on the same side of π' and π, Fig. 20a.

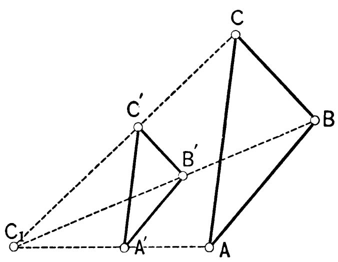


Fig. 20a.

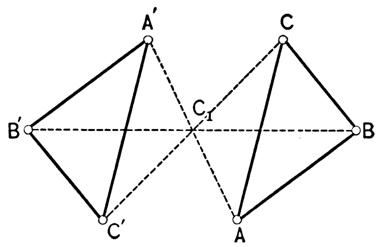


Fig. 20b.

If k is negative, P and P' are on different sides of C, and the center lies between π' and π. If $k = -1$, the center is in the middle of π' and π and the perspective becomes *central symmetry*, Fig. 20b.

C. Affinity.[1]—By this term we designate a perspective whose center is at an infinite distance. All rays through C are parallel and intersect the axis of collineation s at a constant angle. To prove this, draw through every projecting ray, intersecting π' and π in two points P and P', a plane parallel to some fixed plane, and intersecting π' and π in the lines $P'Q$ and PQ; where Q is a point in s, Fig. 21*a*. For all planes of this kind

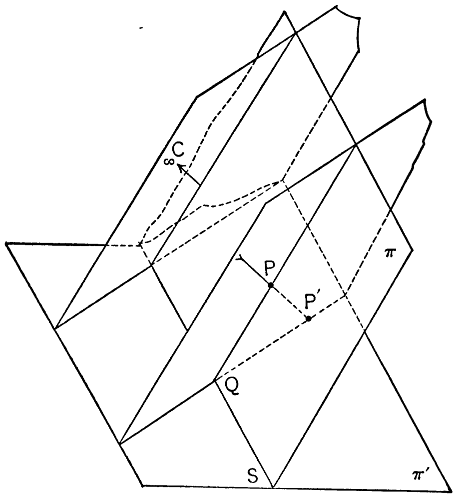


Fig. 21*a*.

the triangles PQP' are similar, i.e., $\frac{P'Q}{PQ}=\text{const.}$; furthermore, for every plane the lines $P'Q$ and PQ include constant angles with s. Hence, after revolving π down into π', Fig. 21*b*, and connecting again P with P', $\angle PQL=\text{const.}$, $\angle P'QL=\text{const.}$, $\frac{P'Q}{PQ}=\text{const.}$; hence $\angle QPL=\text{const.}$, $\angle P'LQ=\text{const.}$, and PP' remains parallel to some direction cutting s at a constant angle.

Now $$PL=\frac{PQ\cdot\sin(PQL)}{\sin(PLQ)}=PQ\cdot\text{const.};$$

[1] Introduced by Möbius in his *Barycentrische Calcul*, p. 150.

similarly $$P'L = \frac{P'Q \sin(P'QL)}{\sin(P'LQ)} = P'Q \cdot \text{const.};$$

hence $$\frac{P'L}{PL} = \text{const.}$$

The same result might be found from Fig. 19, where $\frac{P'L}{PL} = k$. Formulas (II), however, are not valid in this case. To establish the analytical relation between P', (x', y') and P, (x, y) assume now s as the x-axis and any perpendicular to it as the y-axis.

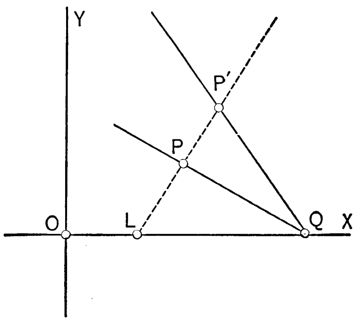


FIG. 21*b*.

Let the constant slope of PP' be m; then the equation of the ray through P is, when ξ and η designate current coordinates, $\eta - y = m(\xi - x)$, and the distance of L from O, Fig. 21*b*, is obtained as $\lambda = \frac{mx - y}{m}$. Now from the figure $\lambda - x' = -k(x - \lambda)$; eliminating λ, there is found

$$\text{(V)} \qquad \begin{cases} x' = x - \dfrac{1-k}{m} y, \\ y' = ky. \end{cases}$$

These are the equations of affinity. Conversely, if a transformation

$$\text{(VI)} \qquad \begin{cases} x' = x + ay, \\ y' = by \end{cases}$$

is given, it may always be represented in the form (V), by putting $b=k$ and $\frac{1-b}{a}=-m$. A characteristic property of this transformation is that closed curves are transformed into closed curves, so that the areas enclosed by the two have the constant ratio k (in V). To prove this assume any triangle ABC and the corresponding triangle $A'B'C'$. Designate the points of intersection

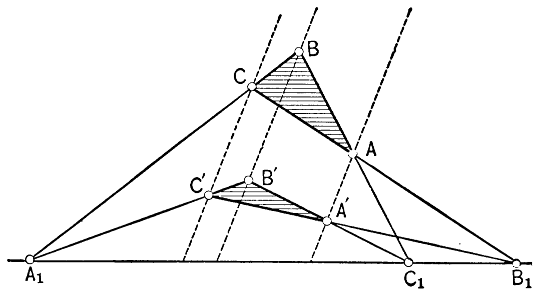


FIG. 22.

of AB and $A'B'$, BC and $B'C'$, CA and $C'A'$ by C_1, A_1, B_1, respectively. Now in Fig. 22

$$(1) \qquad \triangle ABC = \triangle AC_1B_1 + \triangle BA_1C_1 + \triangle CB_1A_1,$$

$$(2) \qquad \triangle A'B'C' = \triangle A'C_1B_1 + \triangle B'A_1C_1 + \triangle C'B_1A_1;$$

but as the corresponding triangles on the left side have equal bases and altitudes of constant ratio k, we have

$$\triangle A'C_1B_1 = k\cdot\triangle AC_1B_1; \quad \triangle B'A_1C_1 = k\cdot\triangle BA_1C_1;$$
$$\triangle C'B_1A_1 = k\cdot\triangle CB_1A_1.$$

Substituting these values in (2) and dividing (1) by (2), there is

$$\triangle A'B'C' = k\cdot\triangle ABC. \qquad \text{Q.E.D.}$$

As these triangles may be infinitesimal, it follows by limiting summations that the same property holds for any corresponding

areas. If k is negative, it follows that the area of $A'B'C'$ is also negative. For $k=-1$

$$\text{(VII)} \qquad \begin{cases} x'=x-\dfrac{2}{m}y, \\ y'=-y, \end{cases}$$

which represents as a special case of affinity *oblique axial symmetry*. For $m=\infty$ this goes over into *orthogonal axial symmetry*. ($x'=x$, $y'=-y$).

When $k=+1$ and $m\neq 0$ an identical transformation is obtained. But the case is also possible where $k=+1$ and $m=0$; i.e., where the rays PP' or AA' are parallel to s, Fig. 23. In

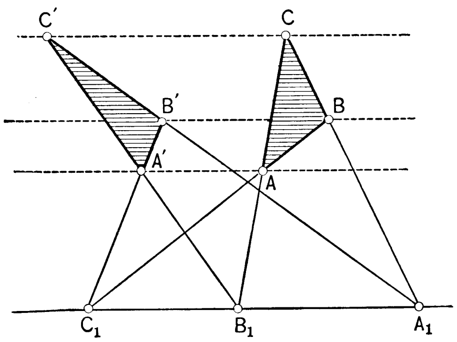


FIG. 23.

this case $\dfrac{1-k}{m}$ can have any value, say $-\lambda$, so that the equations are now

$$\text{(VIII)} \qquad \begin{cases} x'=x+\lambda y, \\ y'=y. \end{cases}$$

The effect of this transformation is that every point is moved parallel to s, and the amount of the motion is proportional to the distance of the point from s. As in (VII), equal areas are

here transformed into equal areas. This transformation is called *elation*.[1]

§ 18. Exercises and Problems.

1. Given a straight line with the equation

$$px+qy+r=0.$$

Find the equation of the perspective of this line, and discuss its position with respect to the original line, q', C, and s.

2. Let $x'y'=f(x')$ be the equation of a curve and suppose that $f(x')$ is not divisible by x' and remains finite for $x'=0$; then for this value of x', y' becomes infinite; in other words, the curve approaches the y-axis asymptotically.

Applying the transformation (II) to this equation, or making a perspective of this curve, its equation becomes

$$a^2xy(dx+ey+f)^{n-2}=\phi(x, y),$$

where $\phi(x, y)$ is a polynomial of x and y of the nth order, provided $f(x)$ is an integral algebraic function of x of the nth degree. For $x'=0$, $y'=\infty$. The corresponding values of x and y are $x=0$ and $y=-\frac{f}{a}$. Thus putting in the above equation $x=0$, the value of y is found to be $-\frac{f}{a}$. *The infinite branch of the curve is therefore transformed into a finite branch.* This is generally true, as it follows directly from equations (II). If $x'=\infty$, $y'=\infty$, $\frac{y'}{x'}=m$ (finite), the corresponding point x, y is necessarily situated in the line r, whose equation is $dx+ey+f=0$. Now $\frac{y}{x}=\frac{y'}{x'}=m$; hence $y=mx$ and from $dx+emx+f=0$, the coordinates $x=-\frac{f}{d+em}$, $y=-\frac{mf}{d+em}$ of the required point are found.

---

[1] Term used by S. Lie, loc. cit.

3. Find the transformed equation of $xy=1$ and discuss it.

4. Find the transformed equation of the circle

$$x^2+y^2=\frac{f^2}{d^2+e^2}.$$

5. Transform perspectively the curve $y'=e^{x'}$. For $x'=\infty$, $y'=\infty$. Now $y=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$; hence $\frac{y'}{x'}=\frac{1}{x'}+1+\frac{x'}{2!}+\frac{x'^2}{3!}+\dots$ and $\lim\limits_{x'=\infty}\left(\frac{y'}{x'}\right)=\infty$. By equations (II),

$$\frac{ay}{dx+ey+f}=e^{\frac{ax}{dx+ey+f}}.$$

Now for $x'=\infty$, $y'=\infty$, we have

$$\frac{y}{x}=\frac{y'}{x'}=\infty \quad \text{and} \quad \frac{x}{y}=0;$$

hence $\dfrac{a}{d\frac{x}{y}+e+\frac{f}{y}}=e^{\frac{a}{d+e\frac{y}{x}+\frac{f}{x}}}$ becomes $\dfrac{a}{e+\frac{f}{y}}=e^0=1;$

hence $y=\dfrac{f}{a-e}$ and $x=0$.

Make the corresponding construction.

6. *A perspective does not change the degree or class of a curve.* As the curve is supposed to be algebraic, we may represent it by the polynomial $f(x, y)=0$, which evidently does not change its degree when transformed. Show this directly.

7. Prove analytically that the transformation

$$x'=x+ay,$$
$$y'=by$$

transforms the area of the triangle ABC $(x_1, y_1;\ x_2, y_2;\ x_3, y_3)$ into an area of the triangle $A'B'C'$ $(x_1', y_1';\ x_2', y_2';\ x_3', y_3')$, so that $\triangle A'B'C'=b\cdot\triangle ABC$. Use determinants.

§ 19. Collineation.[1]

In the last two sections it has been seen that every perspective transformation transforms a straight line into another straight line (into a point if the line passes through the center). The question is now whether there are other transformations with this property. From analytical geometry it is known that *translation* and *rotation* are transformations of this kind.

A. TRANSLATION.—By this operation all points of a plane are moved parallel to a certain fixed direction by the same amount. The equations are

$$\text{(IX)} \qquad \begin{cases} x' = x + a, \\ y' = y + b. \end{cases}$$

The slope of the direction in which (x, y) is moved is

$$\frac{y' - y}{x' - x} = \frac{b}{a},$$

and the amount $\sqrt{a^2 + b^2}$.

B. ROTATION.—If every point (x, y) is turned through an angle ϕ about the center (origin), the coordinates x', y' of the rotated point may be expressed by

$$\text{(X)} \qquad \begin{cases} x' = x \cos \phi - y \sin \phi, \\ y' = x \sin \phi + y \cos \phi. \end{cases}$$

(IX) and (X) are the equations of ordinary motions in a plane. If to a point (x, y) we first apply a rotation (X) and then a translation $x'' = x' + a$, $y'' = y' + b$, and finally writing again x' and y' for x'' and y'', the result is

$$\text{(X}'\text{)} \qquad \begin{cases} x' = x \cos \phi - y \sin \phi + a, \\ y' = x \sin \phi + y \cos \phi + b. \end{cases}$$

[1] Called by CHASLES, in his *Géométrie Supérieure*, art. 99, *homographie; homographic* means to be in collineation. The word collineation goes back to MÖBIUS' *Barycentrische Calcul.*

Such a transformation changes only the position and not the shape of a figure, nor its size. Another transformation which does not change the character of a straight line is

C. Dilatation.—The equations for these are:

$$\text{(XI)} \qquad \begin{cases} x' = \alpha x, \\ y' = \beta y, \end{cases}$$

and may physically be illustrated by stretching a piece of rubber first in the direction of the x-axis and then in the direction of the y-axis. Equal distances along one of the axes are stretched by the same amount. If one of the coefficients α, β is 1, then (XI) represents an orthogonal affinity. Combining the dilatation $x' = \alpha x$, $y' = \beta y$ with the affinity $x'' = x' + \alpha_1 y'$, $y'' = \beta_1 y'$ and then dropping one prime clear through, the result is

$$\begin{aligned} x' &= \alpha x + \alpha_1 \beta y, \\ y' &= \beta \beta_1 y. \end{aligned}$$

Applying to this the rotation $x'' = x' \cos\phi - y' \sin\phi + a'$, $y'' = x' \sin\phi + y' \cos\phi + b'$, and dropping one prime, the result is

$$\begin{aligned} x' &= \alpha \cos\phi \cdot x + (\alpha_1 \beta \cos\phi - \beta\beta_1 \sin\phi) y + a', \\ y' &= \alpha \sin\phi \cdot x + (\alpha_1 \beta \sin\phi + \beta\beta_1 \cos\phi) y + b', \end{aligned}$$

representing thus a combination of a dilatation, an affinity, a translation, and a rotation. Conversely, every transformation

$$\text{(XII)} \qquad \begin{cases} x' = ax + by + c, \\ y' = dx + ey + f, \end{cases}$$

represents such a combination. To prove this, put $a = \alpha \cos\phi$, $d = 2 \sin\phi$; i.e., $\tan\phi = \frac{a}{d}$, $\alpha = \sqrt{a^2 + d^2}$. Further,

$$\begin{aligned} \alpha_1 \beta \cos\phi - \beta\beta_1 \sin\phi &= b, \\ \alpha_1 \beta \sin\phi + \beta\beta_1 \cos\phi &= e, \end{aligned}$$

from which

$$\alpha_1\beta = \frac{b \cos\phi + e \sin\phi}{\cos^2\phi + \sin^2\phi} = b\cos\phi + e\sin\phi = \frac{bd+ae}{\sqrt{a^2+d^2}},$$

$$\beta\beta_1 = e\cos\phi - b\sin\phi \qquad = \frac{ed-ab}{\sqrt{a^2+d^2}},$$

and finally $a' = c$, $b' = f$.

The transformations making up (XII) *all leave the infinitely distant line unchanged and transform areas into proportional areas.* The same is therefore also true of their combination. Such a transformation is called a *Linear Transformation*, or *Linear Deformation.* As all of its constituents are projective, a linear transformation is also projective, i.e., it transforms pencils and ranges into projective pencils and ranges. Perspective and linear transformations are two of the most important projective transformations.

D. General Collineation.—A perspective contains three arbitrary parameters, which is apparent when numerators and denominators of equations (II) are divided by a. Applying to the point x, y a linear transformation

$$x' = ax + by + c,$$
$$y' = dx + ey + f,$$

and then to the transformed point x', y' the perspective

$$x'' = \frac{x'}{d_1x' + e_1y' + f_1},$$

$$y'' = \frac{y'}{d_1x' + e_1y' + f_1},$$

the result is, after dropping one prime,

$$\text{(XIII}') \qquad \begin{cases} x' = \dfrac{ax+by+c}{(d_1a+e_1d)x + (d_1b+e_1e)y + f_1}, \\ y' = \dfrac{dx+ey+f}{(d_1a+e_1d)x + (d_1b+e_1e)y + f_1}. \end{cases}$$

These equations are of the form

$$\text{(XIII)} \qquad \begin{cases} x' = \dfrac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \\[2ex] y' = \dfrac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3}. \end{cases}$$

Conversely, every expression of the form (XIII) can be represented by (XIII′). To show this, put $a_1 = a$, $b_1 = b$, $c_1 = c$; $a_2 = d$, $b_2 = e$, $c_2 = f$; $d_1a + e_1d = a_3$, $d_1b + e_1e = b_3$, $f_1 = a_3$. From

$$d_1a + e_1d = a_3,$$
$$d_1b + e_1e = b_3,$$

we find

$$d_1 = \frac{a_3e - b_3d}{ae - bd} = \frac{a_3b_2 - b_3a_2}{a_1b_2 - b_1a_2},$$

$$e_1 = \frac{b_3a - a_3b}{ae - bd} = \frac{b_3a_1 - a_3b_1}{a_1b_2 - b_1a_2}.$$

By means of these formulas it is possible to resolve every transformation (XIII) into a perspective and into a linear transformation. The principal property of this transformation is that it transforms every straight line and every point projectively into a straight line and a point. It is the general *projective transformation of the plane*, or a *collineation* in the plane.

Dividing numerators and denominators of (XIII) by c_3, it is seen that a collineation generally depends upon EIGHT parameters. To prove that this is the most general transformation which transforms straight lines into straight lines, assume that

$$x' = \frac{P(x, y)}{Q(x, y)},$$

$$y' = \frac{R(x, y)}{S(x, y)}$$

be a transformation of this kind; then the equation of every straight line

$$ax' + by' + c = 0,$$

where a, b, c may have any real values, must be transformed into a linear equation between x and y. Thus

$$aP(x, y) \cdot S(x, y) + bR(x, y)Q(x, y) + cQ(x, y) \cdot S(x, y) = 0$$

must be linear for all real values of a, b, and c. This can only be true if P, Q, R, and S are themselves linear functions of x and y.

§ 20. Geometrical Determination and Discussion of Collineation.

1. The equations of collineation depend upon eight parameters; these, when known, determine a collineation. If, therefore, any four points, of which no three lie in a straight line, are given: $A_1(x_1, y_1)$; $A_2(x_2, y_2)$; $A_3(x_3, y_3)$; $A_4(x_4, y_4)$, we can assume any other four points with the same property as corresponding points of a collineation. That this assumption is legitimate is seen from the eight equations which may be established between the coordinates of the given points A_1, A_2, A_3, A_4 and A_1', A_2', A_3', A_4' by formulas (XIII). The eight independent parameters are now the unknown quantities which from the eight equations of condition may be extracted in a definite manner. Hence the theorem:

There is one and only one collineation which transforms a quadrilateral in a plane into any other quadrilateral of the same plane. Two quadrilaterals in a plane determine a collineation uniquely.

2. An important problem in a collineation is the determination of those elements, points, or straight lines which are not changed in position, i.e., of the *invariant elements*. To find the straight lines which are invariant, assume their equation in the form

$$ax' + by' + c = 0. \tag{1}$$

By the collineation (XIII) this is transformed into the equation

$$(2)\qquad (aa_1+bb_1+cc_1)x+(aa_2+bb_2+cc_2)y+(aa_3+bb_3+cc_3)=0.$$

This will be identical with (1) if the three equations hold $aa_1+bb_1+cc_1=\lambda a$, etc., or

$$(3)\qquad \begin{cases} a(a_1-\lambda)+bb_1 \qquad +cc_1 \qquad =0,\\ aa_2 \qquad +b(b_2-\lambda)+cc_2 \qquad =0,\\ aa_3 \qquad +bb_3 \qquad +c(c_3-\lambda)=0. \end{cases}$$

These are consistent only if the determinant

$$(4)\qquad \begin{vmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_2-\lambda \end{vmatrix}=0,$$

which gives a cubic equation for the proportionality-factor λ. Solving (4) for λ and substituting any of its values in (3), we can easily extract the values of $\frac{a}{c}$ and $\frac{b}{c}$ from any two of equations (3).

These values inserted in (1) give the equation of an invariant straight line of the collineation. As there are three values for λ, there are three such lines. Hence the theorem:

A collineation in a plane leaves a triangle invariant.

The vertices of this triangle are invariant points, while other points of the sides of a triangle are generally transformed into other points of the same sides. (See ex. 6 in § 23.) It may happen that two roots of the determinant (4) are conjugate imaginary, so that the invariant triangle in this case has only one real side and one real point (above example).

3. In equations (XIII) both x' and y' become infinite for all points of the line

$$a_3x+b_3y+c_3=0;$$

hence, in the collineation, to this line corresponds the line at infinity. Solving equations (XIII) for x and y, the result is

$$(5) \qquad \begin{cases} x = \dfrac{(b_2c_3 - b_3c_2)x' + (b_3c_1 - b_1c_3)y' + (b_1c_2 - b_2c_1)}{(a_2b_3 - a_3b_2)x' + (a_3b_1 - a_1b_3)y' + (a_1b_2 - a_2b_1)}, \\ y = \dfrac{(a_3c_2 - a_2c_3)x' + (a_1c_3 - a_3c_1)y' + (a_2c_1 - a_1c_2)}{(a_2b_3 - a_3b_2)x' + (a_3b_1 - a_1b_3)y' + (a_1b_2 - a_2b_1)}. \end{cases}$$

From this it is seen that all points of the line at infinity, $x = \infty$, $y = \infty$ are transformed into the line

$$(6) \qquad (a_2b_3 - a_3b_1)x' + (a_3b_1 - a_1b_3)y' + (a_1b_2 - a_2b_1) = 0.$$

4. Suppose a collineation (XIII) has been applied to a plane. Turn the transformed plane through an angle ϕ about the origin and translate it afterwards in the direction $\tan \theta = \dfrac{b}{a}$ through a distance $\sqrt{a^2 + b^2}$. The result of these successive transformations of the original plane (x, y) is expressed by the equations

$$x'' = \frac{\begin{gathered}(a_1 \cos \phi - a_2 \sin \phi + aa_3)x + \\ (b_1 \cos \phi - b_2 \sin \phi + ab_3)y + c_1 \cos \phi - c_2 \sin \phi + ac_3\end{gathered}}{a_3x + b_3y + c_3},$$

$$y'' = \frac{\begin{gathered}(a_1 \sin \phi + a_2 \cos \phi + ba_3)x + \\ (b_1 \sin \phi + b_2 \cos \phi + bb_3)y + c_1 \sin \phi - c_2 \cos \phi + bc_3\end{gathered}}{a_3x + b_3y + c_3}.$$

The angle ϕ, and a and b, can always be determined in such a manner that

$$\begin{aligned} &b_1 \cos \phi - b_2 \sin \phi + ab_3 = 0, \\ &a_1 \sin \phi + a_2 \cos \phi + ba_3 = 0, \\ &a_1 \cos \phi - a_2 \sin \phi + aa_3 = b_1 \sin \phi + b_2 \cos \phi + bb_3, \end{aligned}$$

so that by this motion (ϕ, a, b) the equations assume the form

$$x'' = \frac{\alpha x + p}{a_3x + b_3y + c_3},$$

$$y'' = \frac{\alpha y + q}{a_3x + b_3y + c_3}.$$

If the original plane is translated in such a manner that $x=x_1-\frac{p}{\alpha}$, $y=y_1-\frac{q}{\alpha}$, the connection between the (x'', y'') plane and the (x_1, y_1) plane will be of the form

$$x''=\frac{\alpha x_1}{\alpha_3 x+\beta_3 y+\gamma_3},$$

$$y''=\frac{\alpha y_1}{\alpha_3 x+\beta_3 y+\gamma_3}.$$

Hence the theorem:

If two collinear planes (figures) are given, it is always possible, by proper motions, to bring both into a perspective position.

And as a corollary:

Any two quadrilaterals in a plane can always be moved into a perspective position; one may be considered as a perspective of the other.

As a special case we have:

Any quadrilateral may be considered as the perspective of a square; and conversely.

On account of its practical importance this proposition will be treated graphically in § 27.

§ 21. Continuous Groups of Projective Transformations.

If to the point x', y' obtained by the projective transformation

$$(1)\qquad \begin{cases} x'=\dfrac{a_1 x+b_1 y+c_1}{a_3 x+b_3 y+c_3}, \\[2ex] y'=\dfrac{a_2 x+b_2 y+c_2}{a_3 x+b_3 y+c_3} \end{cases}$$

another transformation of the same kind

$$(2)\qquad \begin{cases} x''=\dfrac{\alpha_1 x'+\beta_1 y'+\gamma_1}{\alpha_3 x'+\beta_3 y'+\gamma_3}, \\[2ex] y''=\dfrac{\alpha_2 x'+\beta_2 y'+\gamma_2}{\alpha_3 x'+\beta_3 y'+\gamma_3} \end{cases}$$

is applied, the result is

$$(3) \qquad \begin{cases} x'' = \dfrac{A_1x+B_1y+C_1}{A_3x+B_3y+C_3}, \\ y'' = \dfrac{A_2x+B_2y+C_2}{A_3x+B_3y+C_3}, \end{cases}$$

where

$$A_1 = \alpha_1a_1+\beta_1a_2+\gamma_1a_3,\ B_1 = \alpha_1b_1+\beta_1b_2+\gamma_1b_3,\ C_1 = \alpha_1c_1+\beta_1c_2+\gamma_1c_3,$$

$$A_2 = \alpha_2a_1+\beta_2a_2+\gamma_2a_3,\ B_2 = \alpha_2b_1+\beta_2b_2+\gamma_2b_3,\ C_2 = \alpha_2c_1+\beta_2c_2+\gamma_2c_3,$$

$$A_3 = \alpha_3a_1+\beta_3a_2+\gamma_3a_3,\ B_3 = \alpha_3b_1+\beta_3b_2+\gamma_3b_3,\ C_3 = \alpha_3c_1+\beta_3c_2+\gamma_3c_3.$$

Transformation (3) which changes (x, y) directly into (x'', y'') is of the same form as (1) and belongs, therefore, to the totality of all projective transformations. For this reason it is said that all projective transformations of the general type form a *group*. It is *eight-termed* (achtgliederig), since its general equations depend upon eight parameters. In § 20 we saw that every transformation (1) leaves a triangle invariant, and this fact is the characteristic property of the general projective group.

It is not our purpose to discuss all possible projective groups, and we shall simply point out the most important ones.

The six-termed linear transformation

$$(4) \qquad \begin{cases} x' = a_1x+b_1y+c_1, \\ y' = a_2x+b_2y+c_2 \end{cases}$$

is clearly a group. As it is contained in the general group it is called a *six-termed subgroup* of (1) and leaves the line at infinity invariant.

The perspective

$$(5) \qquad \begin{cases} x' = \dfrac{a_1x}{a_3x+b_3y+c_3}, \\ y' = \dfrac{a_1y}{a_3x+b_3y+c_3} \end{cases}$$

is a three-termed subgroup leaving a point (origin $x=0$, $y=0$) and the axis s of collineation (every point of it) invariant. As in these examples, it may be found that *every projective special group leaves a certain figure invariant.* The particular invariant figure is characteristic for the group.

The theory of continuous groups is a creation of SOPHUS LIE [1] and is of the greatest importance in various branches of mathematics, notably in the theory of differential equations.

§ 22. The Principle of Duality.[2]

A. Two forms of projectivity have already been studied (§§ 6, 9, 10, 13)—the projectivity of pencils and that of rays. Two projective pencils generate a curve of the second order; two projective ranges generate a curve of the second class. In the first case the point is the generating element of the curve; in the second it is the straight line. In both cases the equations in point- and line-coordinates are respectively of the second degree.

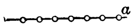
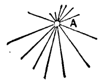
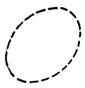
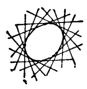
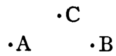
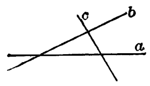
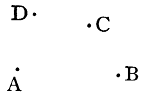
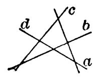
A plane figure may therefore be considered either as a configuration of points or as a configuration of straight lines. This is the principle of *duality*. Two figures are called *dual* if to a point in one corresponds a straight line in the other, and conversely. Below is a scheme of some dual figures, which by the foregoing statements explains itself.

---

[1] *Vorlesungen über continuierliche Gruppen. Theorie der Transformationsgruppen.*

The theory of projective groups has been worked out synthetically by Professor NEWSON and partly by the author himself. See *Kansas University Quarterly*, Vol. V, No. 1.

[2] *Historic Note.*—PONCELET in his Traité, 1822, was the first geometer who showed by his method of reciprocal polars the great importance which dualistic relations have for geometry. GERGONNE, in the Annales de Mathématiques, T. XVI, 1826, stated *the principle of duality* in all its generality and independently of reciprocal polars. PLÜCKER first established the analytic expression for duality, and STEINER gave the equivalent geometric interpretation.

1. Range of points on a straight line. 	Pencils of rays through a point. 
2. Curve. 	Envelope. 
3. Triangle. 	Trilateral. 
4. Quadrangle. 	Quadrilateral. 
5. Points of a plane (∞^2).	Straight lines of a plane (∞^2).
6. Point of a curve.	Tangent of a curve.
7. Tangents from a point to a curve.	Points of intersection of a straight line with a curve.
8. Point of intersection of two straight lines.	Straight line connecting two points.
9. Intersections of curves.	Common tangents of curves.

B. Analytically the principle of duality is expressed by the distinction between point- and line-coordinates: x, y and u, v. If the point with the coordinates x and y satisfies the equation

$$ax+by+c=0,$$

then the point is situated on the straight line which cuts from the x- and y-axes the distances $-\frac{c}{a}$ and $-\frac{c}{b}$. If x and y are kept fixed and a, b, c, or $u=\frac{a}{c}$, $v=\frac{b}{c}$ are varied, then for all values of a, b, c, or u and v, which satisfy $ax+by+c$, or $ux+vy+1=0$, a straight line is obtained which passes through the point (x, y). Hence

$$ux+vy+1=0$$

is the quation of the point (x, y) in line-coordinates u, v. In § 19 the equations for a general collineation were established for

point-coordinates. The problem now is to do the same thing for line-coordinates. A straight line

(1) $$ax' + by' + c = 0$$

by equations (XIII), § 19, is transformed into

$$x(aa_1 + ba_2 + ca_3) + y(ab_1 + bb_2 + cb_3) + ac_1 + bc_2 + cc_3 = 0,$$

or

(2) $$x\frac{aa_1 + ba_2 + ca_3}{ac_1 + bc_2 + cc_3} + y\frac{ab_1 + bb_2 + cb_3}{ac_1 + bc_2 + cc_3} + 1 = 0.$$

Putting $ax' + by' + c$ into the form $ux' + vy' + 1 = 0$ and (2) into the form $u'x + v'y + 1 = 0$, the required equations of collineation in line-coordinates

(XIV) $$\begin{cases} u' = \dfrac{a_1u + a_2v + a_3}{c_1u + c_2v + c_3}, \\[2ex] v' = \dfrac{b_1u + b_2v + b_3}{c_1u + c_2v + c_3}, \end{cases}$$

are obtained. By this transformation every straight line with the coordinates u, v is transformed into a straight line with the coordinates u', v'.

The discussion of these formulas is similar to that of (XIII) and may be left to the student. In (XIII) and (XIV) the analytical expressions for the dualistic interpretation of collineation have been obtained. As for (XIII), the group-property is fundamental for (XIV). In case of perspective in line-coordinates, a_2, a_3, b_2, b_3 vanish from (XIV).

§ 23. Exercises and Problems.

1. Show that the transformation

$$x' = x \cos \phi + y \sin \phi,$$
$$y' = x \sin \phi - y \cos \phi,$$

consists of a rotation through an angle ϕ and a reflection on the x-axis.

2. What becomes of the circle $x^2+y^2=r^2$ after a dilatation; what is the ratio between the enclosed areas before and after dilatation?

3. Investigate the transformation

$$x'=ax+by,$$

$$y'=cx+dy,$$

where $ad-bc=1$.

4. The area included by a closed curve C' in the $x'y'$-plane is obtained by evaluating $\int_c dx'dy'$. If we now transform the $x'y'$-plane by the equations $x'=\phi(x, y)$, $y'=\psi(x, y)$, the area of the transformed curve c is

$$A=\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial x}\right)\int dx\,dy.$$ [1]

Take now a linear transformation

$$x'=ax+by+c,$$
$$y'=dx+ey+f,$$

then the area A' of a curve in the transformed plane expressed in terms of the area A of the original curve is

$$A'=(ae-bd)A,$$

or

$$\frac{A'}{A}=ae-bd;$$

i.e., *in a linear transformation corresponding areas have a constant ratio.*

5. Prove that all points of the xy-plane are transformed into a straight line when $ae-bd=0$.

---

[1] Picard: Traité d'Analyse, Vol. I, pp. 98–102.

6. Find the invariant points from the equations of a general collineation. In these equations set $x=\frac{\xi}{\zeta}$, $y=\frac{\eta}{\zeta}$, $x'=\frac{\xi'}{\zeta'}$, $y'=\frac{\eta'}{\zeta'}$ and designate by λ a factor of proportionality; then

$$(1) \qquad \begin{cases} \lambda\xi' = a_1\xi + b_1\eta + c_1\zeta, \\ \lambda\eta' = a_2\xi + b_2\eta + c_2\zeta, \\ \lambda\zeta' = a_3\xi + b_3\eta + c_3\zeta. \end{cases}$$

If (ξ', η', ζ') is identical with (ξ, η, ζ), we get the condition

$$(2) \qquad \begin{cases} (a_1-\lambda)\xi + b_1\eta + c_1\zeta = 0, \\ a_2\xi + (b_2-\lambda)\eta + c_2\zeta = 0, \\ a_3\xi + b_3\eta + (c_3-\lambda)\zeta = 0, \end{cases}$$

or

$$(3) \qquad \begin{vmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{vmatrix} = 0.$$

This determinant gives three values for λ, consequently from (2) three sets of values for (ξ, η, ζ). Two values of λ, hence *two of the invariant points, may be imaginary, while the line joining them is real* (§ 20, 2).

7. Show that all motions in a plane form a group.

8. Prove the same for affinity;

9. For symmetry (central and orthogonal);

10. Similitude.

11. Find the invariant elements of an affinity.

12. How does a linear equation affect an hyperbola?

§ 24. Orthographic Projection.

1. In an orthographic projection two perpendicular planes of projection are assumed. One is in a horizontal position, and is designated by H; the other in a vertical position, and is designated by V. Both H and V intersect each other in the

ground-line GL and divide the whole space into four quadrants, or angles, Fig. 24. With respect to the observer, space may be described as *above* or *below* H, in *front* or *back* of V. The four angles are now numbered as follows:

I Angle: *above* H, *in front* of V.
II Angle: *above* H, *back* of V.
III Angle: *below* H, *back* of V.
IV Angle: *below* H, *in front* of V.

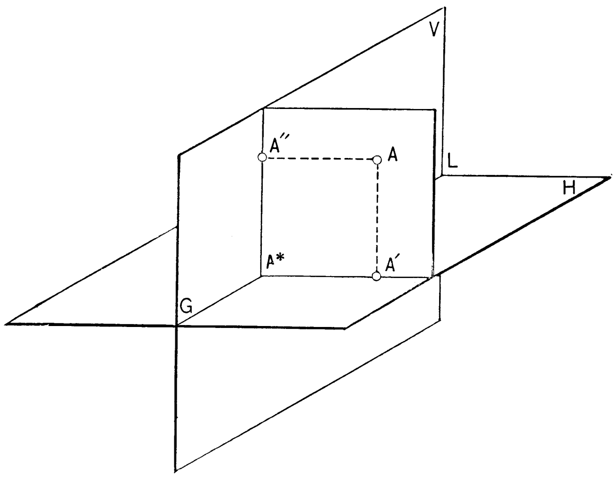


Fig. 24.

In any of these angles, the projections of a point are obtained as the foot-points of the perpendiculars (projectors, projecting lines) from these points to H and V. If A is the point in space, we may designate the horizontal projection of A, which is in H, by A', the vertical projection of A by A'' (in V).

Let the plane through AA' and AA'' intersect GL in A^*, then there is $A''A^*=AA'$, $A'A^*=AA''$. In order to have the representation of these projections in one plane (plane of the drawing; blackboard), one of the planes, for example V, may be rotated about GL, so that the part of V above GL turns from

the observer till it coincides with H. After the rotation the upper portion of V covers the back part of H, and the lower portion of V lies in coincidence with the front part of H. Accordingly the projections of points in the different angles will lie as follows with respect to GL:

Point in	I Angle:	H-projection	*below*,	V-projection	*above* GL;
" "	II " :	"	*above*,	"	*above* " ;
" "	III " :	"	*above*,	"	*below* " ;
" "	IV " :	"	*below*,	"	*below* " .

The same is true of the projections of any figures situated in the different angles.

In Fig. 25 these cases are represented. *The two projections of a point necessarily lie in the same perpendicular (eventually*

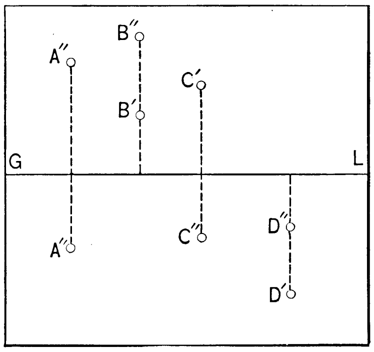


FIG. 25.

extended) to GL. A fixed point in space can be represented in one and only one way by two projections. Conversely, *two points in the same perpendicular to GL always represent the projections of a point (but only one point).*

2. STRAIGHT LINE.—A straight line is determined by two points (the only line or curve determined by two points). *An orthographic projection (of course, also a perspective) of a straight line is also a straight line.* Hence, if A', A''; B', B'' are the projections of two points, the lines joining A', B' and A'', B''

are the corresponding projections of the line AB. Generally any straight line—when produced—pierces H and V. The points where this occurs are called the *traces* of the line and may conveniently be designated by t_1, t_2; s_1, s_2, etc. ($t_1 \equiv$ horizontal, $t_2 \equiv$ vertical trace). *A point P is situated on AB when P' is on $A'B'$, P'' on $A''B''$, and $P'P'' \perp GL$*, Fig. 26. To find the traces t_1, t_2 when $A'B'$ and $A''B''$ are given we notice that the V-projection of a point in H lies in GL, and that the H-projection of a point in V lies also in GL; the other projections coincide with the points themselves, respectively.

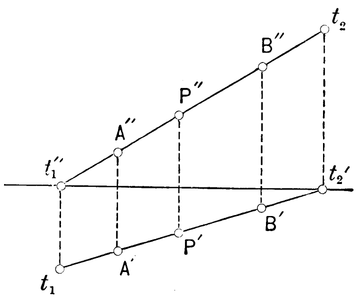


Fig. 26.

Hence, to find the horizontal trace t_1 of AB, produce $A''B''$ till it intersects GL at t_1''. This, being still situated on $A''B''$, is the vertical projection of a point of AB, and as it is also on GL, the point must be in H and is necessarily the required trace t_1. Hence, to find t_1, produce the perpendicular to GL at t_1'' till it intersects $A'B'$ produced at the required t_1. Similarly, t_2 is found by drawing a perpendicular at the point of intersection t_2' of $A'B'$ with GL and producing it till it meets $A''B''$ produced at the required t_2.

It is evident that any two lines l' and l'' may be considered as the projections of a line l. This line is uniquely determined by l' and l''. To prove this draw any two perpendiculars to GL, cutting l' and l'' at A', A'' and B', B''. These points, however, represent the projections of two points and hence the problem is reduced to the foregoing considerations. This proposition is altogether general no matter how the lines may be situated. *If one line is perpendicular to GL, the other reduces to a point.* Usually the projections may be assumed indefinitely extended.

A finite portion may be cut out by two perpendiculars to GL. If the traces t_1 and t_2 are given, the projections are found

by drawing the perpendiculars t_1t_1'' and t_2t_2' to the ground-line; then t_1t_2' is the H-projection, t_2t_1'' the V-projection of the line.

3. PLANE.—A plane is determined

(*a*) By three points not in the same straight line;
(*b*) one point and a straight line;
(*c*) two intersecting lines;
(*d*) two parallel lines.

The most convenient manner to represent a plane is by its *traces*, i.e., its lines of intersection with the planes of projection. *The traces of a plane meet in the ground-line* and may be designated by σ_1, σ_2; τ_1, τ_2; etc. Let S and T be the points where σ_1 and σ_2, τ_1 and τ_2 meet. A plane in a general position extends into all four quadrants, and if nothing else is specified it will be understood that the plane is indefinitely extended.

Ex. 1. Draw the projections of a straight line

(*a*) $\perp H$
(*b*) $\perp V$
(*c*) $\parallel H$
(*d*) $\parallel V$
(*e*) $\parallel GL$
(*f*) $\perp GL$
(*g*) in a plane $\perp GL$
(*h*) cutting GL
(*i*) in H or V
(*j*) in V and $\perp H$,

and repeat the construction in all four angles; also construct the traces in every case.

Ex. 2. Draw the traces of a plane

(*a*) $\perp H$
(*b*) $\perp V$
(*c*) $\perp H$ and V
(*d*) $\parallel GL$
(*e*) $\parallel H$
(*f*) $\parallel V$
(*g*) passing through GL
(*h*) solve the foregoing problems in all four angles.

Ex. 3. A *profile plane* is a plane $\perp GL$. Given a plane $\parallel GL$; find its distance from GL.

Ex. 4. Given any plane; locate a point in this plane; i.e., draw its projections.

Ex. 5. *A straight line lies in a plane when its traces lie in the corresponding traces of the plane; conversely, a plane passes through*

a line if its traces pass through the corresponding traces of the line. Construct the traces of the planes determined by (a), (b), (c), or (d) under 3, § 24.

§ 25. Affinity between Horizontal and Vertical Projections of a Plane Figure.

RABATTEMENT.

1. The orthographic projections of a figure in a plane $\sigma_1 S \sigma_2$ are perspectives with infinitely distant centers in directions perpendicular to H and V. In this case any of the projections and the corresponding revolved figure are in the relation of affinity (orthogonal affinity with one of the traces as an axis). But there exists also affinity between horizontal and vertical projections of a plane figure, as we shall now see.

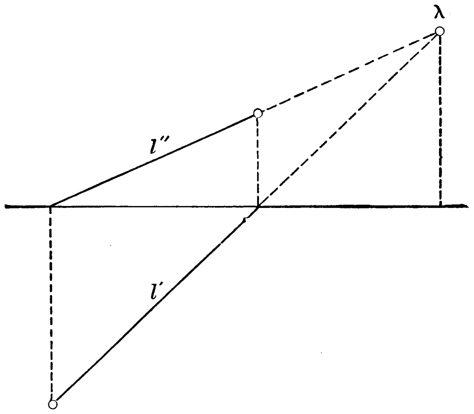


FIG. 27.

2. Let P be the bisecting plane of the first and third angles, Q the bisecting plane of the second and fourth angles. *The projections of a point in P are equally distant from GL; those of a point in Q coincide.*

Hence, to find the point of intersection of any line l with Q, produce l' and l'' till they intersect at λ; λ represents the coinciding

projections of the required point of intersection, Fig. 27. If this point λ is infinitely distant, i.e., if $l' \parallel l''$, then l is parallel to Q.

If a plane P is given, then all lines in this plane will generally intersect the line of intersection s of P and Q. Hence the corresponding projections of all lines in P will meet in points of

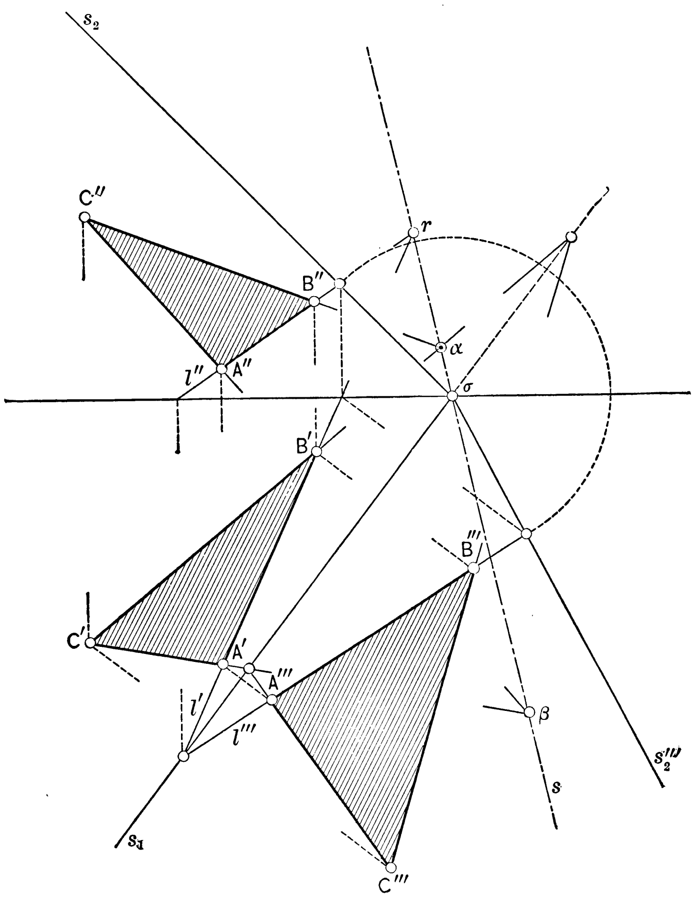


Fig. 28.

s', s''; this line is therefore the axis of affinity which exists between the projections of figures in a plane P. If A is a point in P (traces s_1, s_2), then we can rabat P about s_1 into H. During

the rabattement A describes a circle with center in s_1; A' describes a perpendicular to s_1. A line l, its horizontal projection l', and its rabatted position l''' meet in the same point of s_1, Fig. 28. Hence there exists also affinity between the horizontal projection of a plane figure and its rabattement into H. In Fig. 28 these affinities are illustrated in case of a triangle. If the horizontal projection of a triangle, $A'B'C'$ and A'' are given, B'' and C'' may be found by applying the principle of affinity. Thus $A'B'$ meets $A''B''$ in a point r of s (s', s''), and B', B'' lie in a perpendicular to GL; hence join A'' with r and through B' draw $B'B'' \perp GL$, thus determining B'' Similarly C'' may be obtained. By the same principle the rabattement $A'''B'''C'''$ may be constructed if one point, say A''', is known. An interesting special case is obtained when s_1 and s_2 are equally inclined towards GL. Then s is perpendicular to GL and corresponding projections of closed figures have equal areas, Fig. 29. This case has been discussed in § 17, Fig. 23, *elation*.

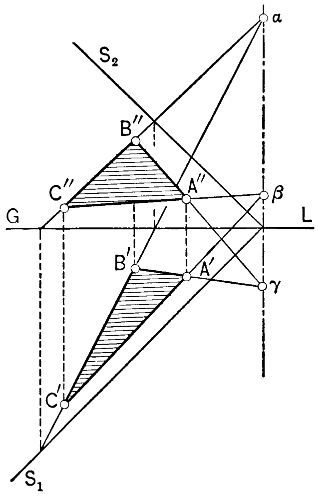


FIG. 29.

Ex. 1. Discuss the affinity which results when s_1 coincides with s_2.

Ex. 2. What must be the position of a plane P to obtain *orthogonal* affinity, § 17?

Ex. 3. What plane gives orthogonal symmetry?

Ex. 4. What is the position of a plane if the projections of any point in this plane are always equally distant from the axis of affinity s? (Oblique symmetry.) Make use of a profile-plane.

Ex. 5. *A straight line is perpendicular to a plane if its projections are perpendicular to the corresponding traces of the plane.* Prove this proposition.

Ex. 6. What is the position of a plane whose traces coincide?

§ 26. Homologous Triangles.

1. In § 15, treating of central projection, two planes π and π' intersecting each other in s and a center of projection $V(C)$ were assumed. Consider now any triangle ABC in π and find its projection $A'B'C'$ in π'; then a pyramid with base $A'B'C'$ and vertex V is obtained which by the plane π is cut in the triangle ABC. Revolving π and with it ABC about s down into π', thus assuming the position $A_1B_1C_1$, then from the laws of perspective we know that $A'A_1$, $B'B_1$, $C'C_1$ are concurrent at a point W, and the points of intersection of $A'B'$ and A_1B_1, $B'C'$ and B_1C_1, $C'A'$ and C_1A_1 are collinear, i.e., lie on the same straight line. Triangles with this property are called *homologous*,[1] Fig. 30. Any two triangles for which the lines joining corresponding vertices are concurrent may always be considered as resulting by the foregoing projection and rabattement, so that the following theorem holds:

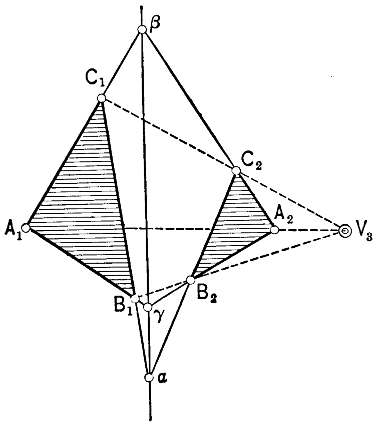


FIG. 30.

If two triangles $A_1B_1C_1$ and $A_2B_2C_2$ are situated in such a manner that the lines joining corresponding points like A_1A_2, B_1B_2, C_1C_2 are concurrent at V_3, then the points of intersection of corresponding sides, in symbols $\left.\begin{matrix}A_1B_1\\A_2B_2\end{matrix}\right\}\gamma$, $\left.\begin{matrix}B_1C_1\\B_2C_2\end{matrix}\right\}\alpha$, $\left.\begin{matrix}C_1A_1\\C_2A_2\end{matrix}\right\}\beta$, *are collinear on s.*

[1] CASEY in his sequel to Euclid uses the word *homologous.* Mr. LEUDESDORF in his translation of Cremona's Projective Geometry, p. 10, uses *homological.* See also PONCELET, loc. cit.

Conversely:

If two triangles are situated in such a manner that the points of intersection α, β, γ of corresponding sides are collinear, then the lines joining corresponding points are concurrent.

It is noticed that these theorems are simply a specialization of the general laws of central projection or perspective. They include evidently the laws of affinity as special cases (infinite point of concurrence, plane intersections of triangular prisms, orthographic and generally parallel projections).

2. THEOREM.—*The centers of homology of three homologous triangles with the same axis of homology are collinear.*

Let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be the three triangles whose corresponding sides meet in the collinear points α, β, γ. Consider the two triangles, Fig. 31, $A_1A_2A_3$ and $B_1B_2B_3$, then it is

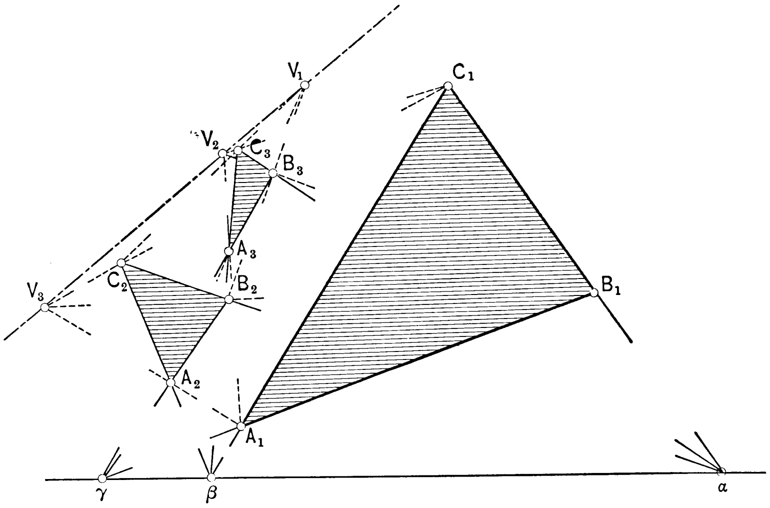


FIG. 31.

seen that the lines A_1B_1, A_2B_2, A_3B_3 are concurrent at γ. Hence the intersections of their corresponding sides are collinear; but these points, V_1, V_2, V_3, are the centers of homology of the given three triangles, Q.E.D.

COROLLARY.—*The three triangles* $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$

have the same axis of perspective; and their centers of homology are the points α, β, γ. *Hence the centers of homology of these triangles lie on the axis of homology of the triangles* $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, *and conversely.*

3. THEOREM.—*The three axes of homology of three homologous triangles with the same center of homology are concurrent.*

Let $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ be the three triangles with the common center of homology V, Fig. 32. Consider the two

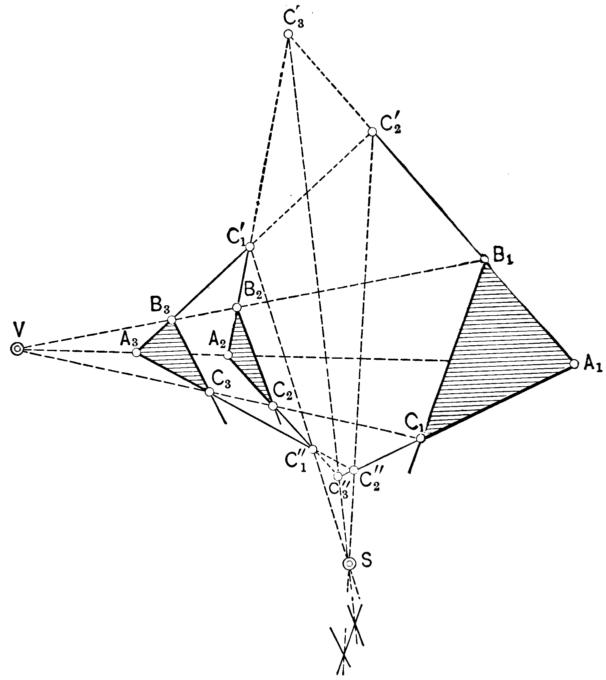


FIG. 32.

triangles formed by the two systems of lines A_1B_1, A_2B_2, A_3B_3 and A_1C_1, A_2C_2, A_3C_3. These two triangles $C_3'C_1'C_2'$ and $C_3''C_1''C_2''$ are in perspective, the line $VA_1A_2A_3$ being their axis of homology. Hence the lines joining their corresponding vertices are concurrent at S, which proves the theorem.[1]

[1] See CASEY'S *Sequel to Euclid*, ed. 1900, pp. 77–88. Casey's proofs are based exclusively upon metrical properties.

4. THEOREM.—*If in two complete quadrilaterals five pairs of corresponding sides intersect in five collinear points, the point of intersection of the sixth pair will be collinear with these.*

Suppose that AB, $A'B'$; BC, $B'C'$; CD, $C'D'$; DA, $D'A'$; BD, $B'D'$ are the pairs of sides intersecting on a fixed line s, Fig.

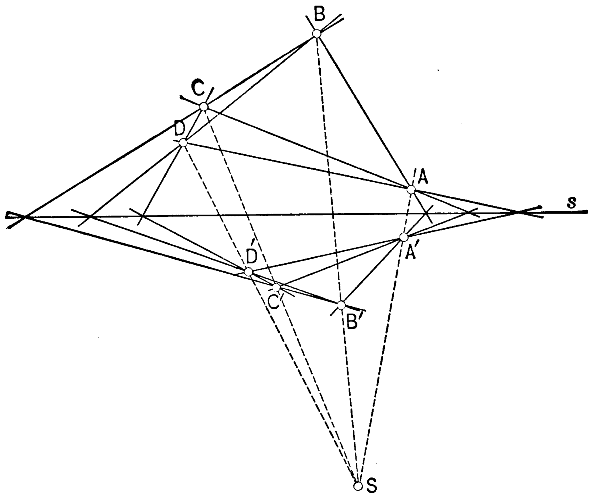


FIG. 33.

33. Now ABC and $A'B'C'$ are homologous triangles, consequently AA', BB', CC' pass through a fixed point S. Also BCD and $B'C'D'$ are homologous, and as BB' and CC' pass through S, DD' also passes through S. Hence, as AA', CC', DD' pass through S, the triangles ACD and $A'C'D'$ are also homologous. Now AD and $A'D'$, CD and $C'D'$ intersect in points of s. Consequently also AC and $A'C'$ intersect on s. This, however, is the sixth pair, Q.E.D. The line of collinearity may, of course, be infinitely distant.

Ex. 1. Prove dualistically: If, in two complete quadrangles, lines joining five pairs of corresponding points are concurrent, the line joining the sixth pair is concurrent with these.

Ex. 2. Consider any three spheres in space which do not intersect and exclude each other. Let c_1, c_2, c_3 be their centers

and construct their external common tangent-cones with the vertices E_1, E_2, E_3. A common tangent-plane to the cones $(E)_1$, (E_2) is necessarily also tangent to (E_3); hence (E_1), (E_2), (E_3) have the same two external common tangent-planes and E_1, E_2, E_3 are therefore necessarily collinear with the line of intersection of these two planes. Similarly it is seen that the internal common tangent-cones of (E_1) and (E_2) and of (E_2) and (E_3) have two common tangent-planes which are also common to the external tangent-cone of (E_1) and (E_3). Hence, designating the vertices of the internal common tangent-cones by I_1, I_2, I_3, we have the following triads of collinear points:

$$E_1E_2E_3, \quad E_1E_2I_3, \quad E_2E_3I_1, \quad E_3E_1I_2;$$

i.e., *the points of similitude of three spheres in space form a complete quadrilateral.*

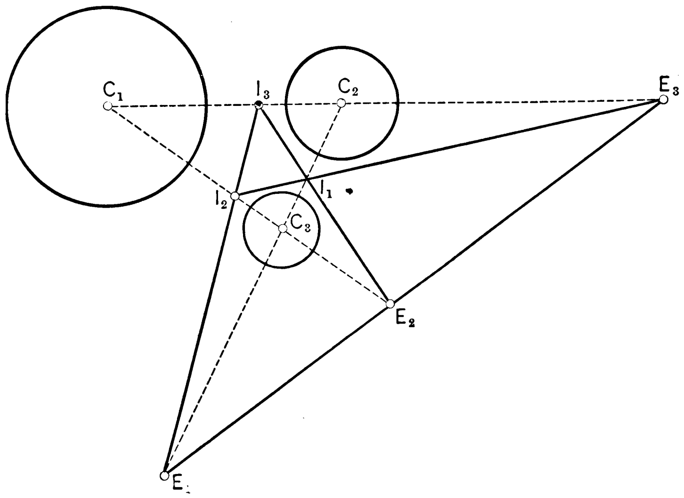


FIG. 34.

In any orthographic projection the spheres are projected into circles and the E's and I's into their centers of similitude; and since any three circles in a plane may always be considered as the projections of three spheres in space excluding one another,

the foregoing proposition also holds for three circles in a plane, Fig. 34.

Ex.. 3. If we now take four spheres in space with the centers C_1, C_2, C_3, C_4, and designating the external and internal centers of similitude respectively by E_{ik}, I_{ik} for the spheres with the centers C_i and C_k, then we find that *all external centers of similitude lie in a plane.* To prove this, remark that there are six centers of this kind, E_{12}, E_{23}, E_{34}, E_{41}, E_{13}, E_{24}, and that these are arranged in groups of three on six straight lines; they form, therefore, a complete quadrilateral and are coplanar, Fig. 35. The centers

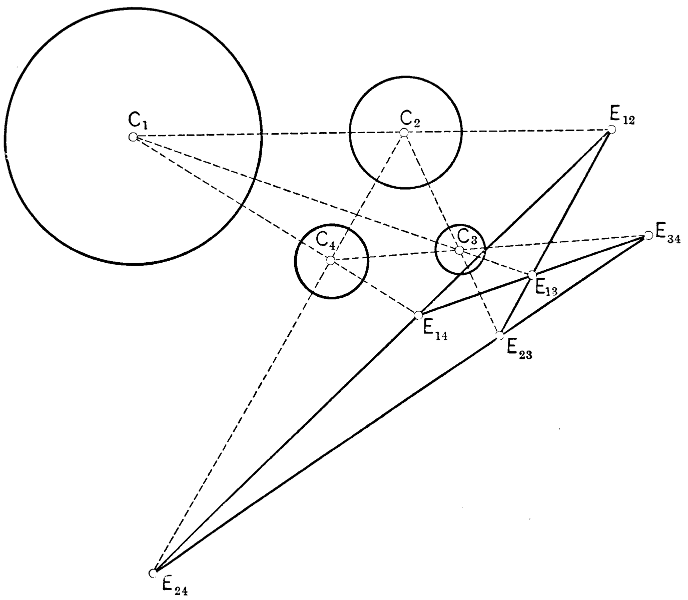


FIG. 35.

of similitude of three spheres are always coplanar; and since four groups of three out of four spheres may be formed, four more quadrilaterals of points of similitude may be formed. The twelve points of similitude are distributed three by three on sixteen axes of similitude. Of the latter, four pass through every point of similitude.

If $pqrs$ designates each arrangement of the numbers 1 2 3 4, (pq) the external, $(\overline{pq})$ the internal point of similitude of Cp and Cq; moreover, if (pqr) is the axis of similitude passing through (pq), (pr), (qr), finally $(pq\bar{r})$ the one through (pq), $(\overline{pr})$, $(\overline{qr})$, then the axes may be represented by the following table:

(234)	$(13\bar{4})$	$(1\bar{2}4)$	$(12\bar{3})$
$(1\bar{3}4)$	$(\bar{2}34)$	(123)	$(\bar{1}24)$
$(12\bar{4})$	$(\bar{1}23)$	$(2\bar{3}4)$	(134)
$(1\bar{2}3)$	(124)	$(\bar{1}34)$	$(23\bar{4})$

In this table two axes which belong neither to one and the same line, nor to the same column, have always a point of similitude in common, while this is not true of two axes belonging to the same column or line.

This configuration, which was discovered by Poncelet, is now known as *Reye's configuration.*[1]

§ 27. A Few Applications to Perspective.

1. Perspective of a Square.—In § 20 it was seen that there is always a collineation transforming a quadrilateral into any other quadrilateral. The proof of this proposition was analytical. In view of its practical application the special case is of interest:

Every quadrilateral may be considered as the central projection or perspective of a rectangle or of a square.

Let $A'B'C'D'$ be any quadrilateral and $L'M'N'$ its diagonal points, Fig. 36. If this is the central projection of a rectangle, the line joining M' with N' must be the vanishing line q', and M' and N' are the vanishing points of the two pairs of parallel sides. From this it is clear that the center of perspective joined with M' and N' gives two perpendicular lines. In other words, the center is situated on a circle having $M'N'$ as a diameter

[1] See *Archiv für Mathematik und Physik*, 3d series, Vol. I, pp. 124–132.

(see § 15). Any point on this circle as a center and any line $\parallel q'$ as an axis determine a perspective in which the original quadrilateral is the perspective of a rectangle. This rectangle can be constructed without difficulty. The quadrilateral may also be considered as the perspective of a square. The center O must now also be situated on a circle over $E'F'$ as a diameter. O is therefore the point, or one of the points, of intersection of

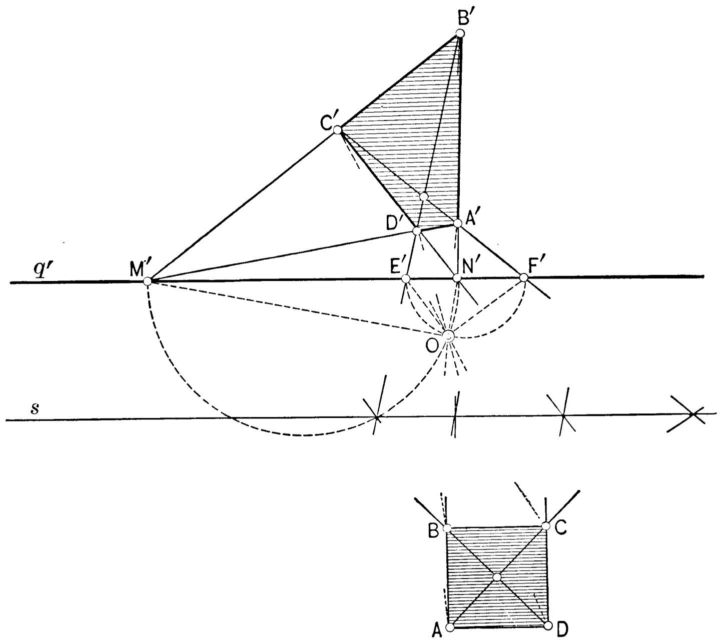


FIG. 36.

the circles over $M'M'$ and $E'F'$ as diameters. In § 8 the harmonic properties of the complete quadrilateral were obtained analytically. Constructing the diagonals of a square, of which one is at an infinite distance, those properties appear immediately from the square, and as a projection, does not change a cross-ratio, it is evident that the same harmonic properties hold for the complete quadrilateral.

2. PERSPECTIVE OF CIRCLES.—Every perspective of a circle is called a *conic section* or simply *conic*. If two lines are tangent to each other at a point A, then the perspectives of these lines are tangent at the perspective A' of A. Hence if a circle is inscribed to a square, the perspective will give a conic inscribed to a quadrilateral.

The problem of drawing perspectives of circles may therefore be reduced to the problem of inscribing conics to quadrilaterals.

By this method the problem can be solved in a simpler manner than by the ordinary construction from the given circle and the data of perspective. In Fig. 37 a square $PQRS$ and the inscribed circle with the points of tangency AA_1 and BB_1 have been assumed. Divide OB and BQ and BP into the same numbers of equal parts and number them from O to B and from P and Q towards B, starting every time with o. Connect A with any of the division-points on BP, and A_1 with the corresponding point on OB. The point of intersection K of these two rays is

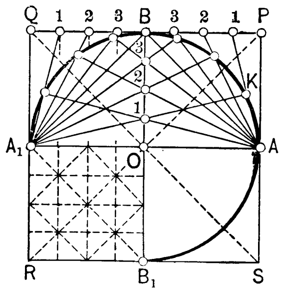


FIG. 37.

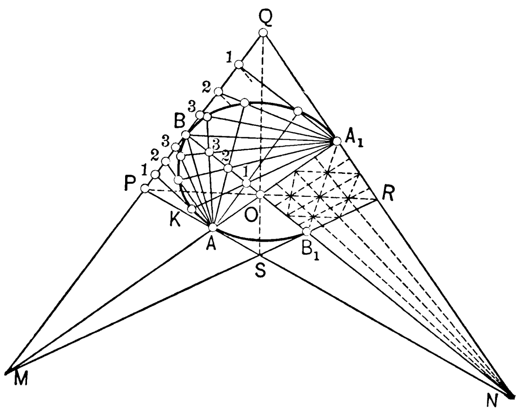


FIG. 38.

a point of the circle. In a similar manner the points of the circle in the remaining quadrants may be located. Now the

rays from A and A_1 form two projective pencils, and their product is therefore a curve of the second order. As $\angle A_1KA$ is a right angle, this curve is a circle (indeed $\triangle A_1O1 = \triangle AP1$, hence $\overline{A_1 1} \perp \overline{A1}$) and is therefore identical with the assumed circle.

Now, in order to inscribe a conic into the quadrilateral $PQRS$, touching at AA_1 BB_1, Fig. 38, construct the point O as the intersection of the diagonals PR and QS. Joining O to M and N and producing gives AA_1 BB_1. Applying the same principle of bisection by diagonals in analogy with Fig. 37, the proper division on OB and PQ is obtained. Having these, the inscribed conic is found in exactly the same manner as the inscribed circle. The proof of this construction is evident, since every quadrilateral may be considered as the perspective of a square, and the perspective does not destroy the projectivity of pencils.

This construction is very effective in perspective drawing, being applicable to all kinds of quadrilaterals.

It is also remarked that conics (perspectives of circles) are curves of the second order.

This idea will be fully discussed in the next chapter.

§ 28. Exercises and Problems.

1. Given two straight lines and a point; to draw a straight line through this point passing through the inaccessible point of intersection of the given lines.

SOLUTION.—Let g and h be the given lines and A the given point. Through A draw any two lines cutting g and h in PQ and RS, Fig. 39*a* and 39*b*. Join PS and QR and produce till they intersect at M. Designating the inaccessible point by N,

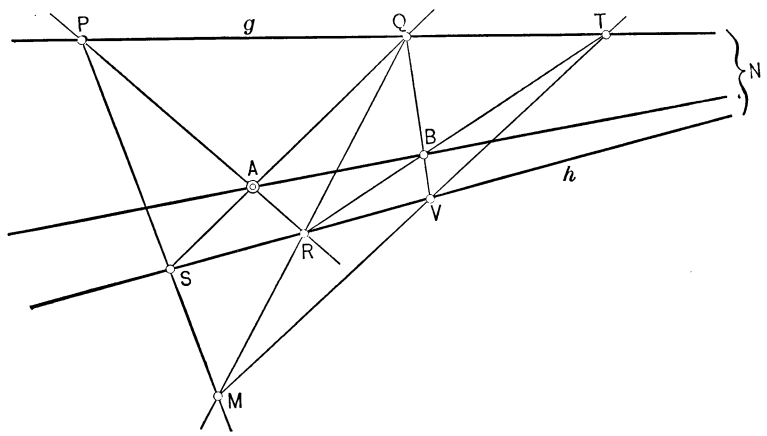


FIG. 39*a*.

$PQRS$ may be considered as the perspective of a square having M, N, A as diagonal points. Hence any third line through M cutting g and h at T and V is the perspective of a line parallel to QR and PS. From this it follows that the line joining A to the point of intersection B of TR and QV is the required line.

2. Inscribe a conic within a rectangle; within a trapezium; a rhombus.

3. Draw two homologous quadrilaterals.

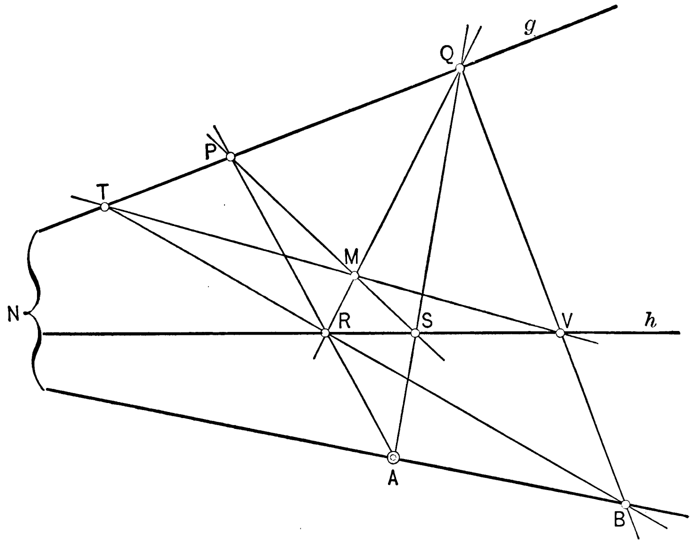


FIG. 39*b*.

4. Draw the perspective of a cube; of a cylinder; of a pyramid; of a hexagonal prism.

5. Draw the perspective of two concentric circles.

CHAPTER III.

THEORY OF CONICS.

§ 29. Introduction.

The Greeks originally studied conics as plane sections of cones.[1] Steiner and Chasles considered them as products of projective pencils and ranges, defined by anharmonic ratios. von Staudt and Reye, however, define this relation purely by harmonic division. I shall follow Steiner's method, by which the projective properties of the circle (see § 12) are easily obtained and transformed to conics by central projection. Following this method, it becomes necessary to show that all curves of the second degree as obtained by projective pencils and ranges are also produced by plane sections of cones, or as perspective collineations of the circle.

Conversely, it must be shown that every curve of the second degree may be projected into a circle. This is the method followed by Poncelet, Steiner, and a majority of modern writers on projective geometry. From a purely geometrical standpoint von Staudt's and Reye's methods are to be preferred.

---

[1] MENACHMUS obtained conics as intersections of planes perpendicular to the elements of a right cone. In case of an "acute-angled cone" (opening angle at the vertex $< 90^\circ$) the conic was called ellipse; of a "right-angled cone" (angle at vertex $= 90^\circ$) a parabola; of an "obtuse-angled cone" (angle at vertex $> 90^\circ$) an hyperbola. APOLLONIUS, who introduced these names, extended the proofs to oblique cones.

§ 30. Identity of Curves of the Second Order and Class and Conics.

The general equation of a circle is

$$(1) \qquad (x-a)^2+(y-b)^2=r^2.$$

To obtain the equation of this circle in line-coordinates, assume the equation of its tangent in the form

$$(2) \qquad ux+vy+1=0.$$

This represents a tangent if the distance of the center (a, b) from the line (2) is r; i.e., if

$$\frac{au}{\sqrt{u_2+v_2}}+\frac{bv}{\sqrt{u_2+v_2}}+\frac{1}{\sqrt{u_2+v^2}}=r,$$

or

$$(3) \qquad r^2(u^2+v^2)-(au+bv+1)^2=0.$$

Every pair of values u, v satisfying (3) gives the line-coordinates of a straight line tangent to the circle (1). Equation (3) represents, therefore, (1) in line-coordinates (see § 6). Both (1) and (3) depend upon three essential parameters. The formulas for perspective in their dual interpretation each depend upon three essential parameters. Hence, applying a perspective to either (1) or (3), i.e., to the given circle, we can in both cases dispose of the six parameters in such a manner that the transformed equations assume any given form of the second degree in x and y, or in u and v. This means that *every curve of the second order or class may be considered as a conic section (perspective of a circle).*

Conversely, if the general equation of a conic of the second degree is given, it is always possible to find a perspective which will transform this equation into that of a circle. Hence *every curve of the second degree may be projected perspectively into a circle. Curves of the second degree and conics are therefore identical.*

Ex. 1. The perspective transformation

$$x' = \frac{x}{lx + my + n},$$

$$y' = \frac{y}{lx + my + n}$$

transforms the general equation

$$ax'^2 + by'^2 + cx'y' + 2dx' + 2ey' + f = 0$$

into

$$x^2(a + 2dl + fl^2) + y^2(b + 2em + fm^2) + xy(c + 2dm + 2el + 2flm) + \\ x(2dn + 2fln) + y(2en + 2fmn) + fn^2.$$

Find the values of l and m which will transform the given equation into that of a circle.

Ex. 2. Solve the dual problem of Ex. 1.

Ex. 3. Find a circle and a perspective, so that the perspective of the circle is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Ex. 4. Given the circle $x^2 + y^2 = 1$. Transform this circle by the perspective

$$x = \frac{x'}{y' - 1}, \quad y = \frac{y'}{y' - 1}.$$

Discuss the result geometrically and show that the center of the circle is the focus of the transformed circle.

§ 31. Linear Transformation of a Curve of the Second Order.

1. By the translation

$$(1) \qquad \begin{cases} x = x' + a_1, \\ y = y' + b_1 \end{cases}$$

the equation

(2) $$ax^2+2bxy+cy^2+2dx+2ey+f=0$$

is transformed into

$$ax'^2+2bx'y'+c'y^2+2(aa_1+bb_1+d)x'+2(ba_1+cb_1+e)y'+ aa_1^2+2ba_1b_1+cb_1^2+2da_1+2eb_1+f=0.$$

In order that the coefficients of x' and y' disappear, a_1 and b_1 must be chosen, so that

$$aa_1+bb_1+d=0,$$
$$ba_1+cb_1+e=0.$$

If $ac-b_2\neq 0$, we find for a_1 and b_1 the values

(3) $$a_1=\frac{be-cd}{ac-b^2}, \quad b_1=\frac{bd-ae}{ac-b^2}.$$

By this assumption the transformed equation reduces to (the x' and y' being replaced by x and y)

(4) $$ax^2+2bxy+cy^2+\frac{\varDelta}{ac-b^2}=0,$$

where

$$\varDelta=d(be-cd)+e(bd-ae)+f(ac-b^2)=\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$$

2. Turning the coordinate axes now through an angle θ; i.e., making the transformation

(5) $$\begin{cases} x=x'\cos\theta-y'\sin\theta, \\ y=x'\sin\theta+y'\cos\theta, \end{cases}$$

and again suppressing the primes of x and y, the result is

$$Ax^2+2Bxy+Cy^2+\frac{\varDelta}{ac-b^2}=0,$$

where

$$\begin{aligned} A &= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta, \\ 2B &= (c-a)\sin 2\theta + 2b\cos 2\theta, \\ C &= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta. \end{aligned}$$

Choosing θ so that $B=0$, or

$$(6) \qquad \tan 2\theta = \frac{2b}{a-c},$$

the transformed equation reduces to

$$(7) \qquad Ax^2 + Cy^2 = \frac{\Delta}{b^2 - ac}.$$

To determine A and C, we have from the foregoing expressions for A, $2B$, C:

$$(8) \qquad \begin{cases} A + C = a + c, \\ B^2 - A\cdot C = b^2 - ac. \end{cases}$$

But when θ is chosen so that $B=0$, $A\cdot C = ac - b^2$. Hence A and C are roots of the equation

$$z^2 - (a+c)z + ac - b^2 = 0.$$

If now $b^2 - ac \neq 0$, two cases, $b^2 - ac > 0$ and $b^2 - ac < 0$, must be distinguished.

It is further assumed that $\Delta \neq 0$. In the first case, $b^2 - ac > 0$, it follows, since now $AC = ac - b^2 < 0$, that A and C are of different sign. Hence, no matter what the sign of Δ, $\dfrac{\Delta}{A(b^2-ac)}$ and $\dfrac{\Delta}{C(b^2-ac)}$ have different signs and are always real. The equation therefore represents an hyperbola. If, in addition to $b^2 - ac > 0$, $\Delta = 0$, then the equation may be resolved into two linear factors; the hyperbola degenerates into two intersecting straight lines.

In the second case, $b^2 - ac < 0$, $A\cdot C = ac - b^2 > 0$. Both A and

C have the same sign; hence also $\frac{\Delta}{A(b^2-ac)}$ and $\frac{\Delta}{C(b^2-ac)}$ have the same sign. According as this is positive or negative, the equation represents a real or an imaginary ellipse. If $\Delta=0$, the ellipse degenerates into two intersecting imaginary lines.

3. Finally the case $b^2-ac=0$ must be considered. Here $b=\pm\sqrt{ac}$. Considering the case $b=+\sqrt{ac}$, the general equation (2) reduces to (coordinates x', y')

$$(9) \qquad (\sqrt{a}x'+\sqrt{c}y')^2+2dx'+2ey'+f=0.$$

Putting $\sqrt{a}=r\sin\theta$, $\sqrt{c}=r\cos\theta$; i.e., $r^2=a+c$, $\tan\theta=\sqrt{\frac{a}{c}}$, and dividing the whole equation by r^2, we have

$$(10) \qquad (x'\sin\theta+y'\cos\theta)^2+\frac{1}{r^2}(2dx'+2ey'+f)=0.$$

Turning the coordinate axes through an angle θ; i.e., putting

$$x=x'\cos\theta-y'\sin\theta,$$
$$y=x'\sin\theta+y'\cos\theta,$$

or

$$x'=x\cos\theta+y\sin\theta,$$
$$y'=-x\sin\theta+y\cos\theta,$$

equation (10) becomes, after putting $\sin\theta=\frac{\sqrt{a}}{\sqrt{a+c}}$, $\cos\theta=\frac{\sqrt{c}}{\sqrt{a+c}}$, and reducing,

$$(11) \qquad y^2+\frac{2}{(a+c)\sqrt{a+c}}\left\{(d\sqrt{c}-e\sqrt{a})x+(d\sqrt{a}+e\sqrt{c})y+\tfrac{1}{2}f\sqrt{a+c}\right\}=0.$$

Making finally the translation, by replacing y and x by $y+\beta$ and $x+\alpha$ respectively, we have

(12) $$y^2+\frac{2}{(a+c)\sqrt{a+c}}\Big\{(d\sqrt{c}-e\sqrt{a})x+ (d\sqrt{a}+e\sqrt{c}+\beta(a+c)\sqrt{a+c})y+\alpha(d\sqrt{c}-e\sqrt{a})+ \frac{\beta^2}{2}(a+c)\sqrt{a+c}+\tfrac{1}{2}f\sqrt{a+c}\Big\}.$$

Letting $\beta=\dfrac{e\sqrt{c}+d\sqrt{a}}{(a+c)\sqrt{a+c}}$ and

$$\alpha=-\frac{(e\sqrt{c}+d\sqrt{a})^2+f(a+c)^2}{2(d\sqrt{c}-e\sqrt{a})(a+c)\sqrt{a+c}},$$

(12) becomes

$$y^2=-2\frac{d\sqrt{c}-e\sqrt{a}}{(a+c)\sqrt{a+c}}x,$$

which represents a parabola. When $b^2-ac=0$, then

$$\Delta=\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}=-(cd^2-2bde+ae^2).$$

But $b=+\sqrt{ac}$, hence

$$\Delta=-(ae^2-2\sqrt{ac}\cdot de+cd^2)=-(e\sqrt{a}-d\sqrt{c})^2;$$

hence

(13) $$y^2=2\frac{\sqrt{-\Delta}}{(a+c)\sqrt{a+c}}x.$$

If in (11) $d\sqrt{c}-e\sqrt{a}=0$, i.e., $\Delta=0$, then the equation may be resolved into two linear factors in y only, and represents consequently two parallel lines. These are real and distinct, coincident (real), imaginary and distinct, according as

$$(d\sqrt{a}+e\sqrt{c})^2-f(a+c)^2 \gtreqless 0.$$

The case $b=-\sqrt{ac}$ may be treated in a similar manner and leads to no new result.

4. From the foregoing short discussion it is seen that the character of the general equation of the second order may be established by means of translations and rotations; i.e., by special

cases of the linear transformation. In this, two algebraic expressions between the coefficients are of fundamental importance, namely, $b^2 - ac = \begin{vmatrix} b & c \\ a & b \end{vmatrix}$, which for abbreviation we may designate by τ, and

$$\varDelta = d(be - cd) + e(bd - ae) + f(ac - b^2) = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$$

According as $\tau \gtreqless 0$ the general equation represents an hyperbola, a parabola, or an ellipse, if $\varDelta \neq 0$. When $\varDelta = 0$ these curves degenerate into intersecting, parallel or coincident, or imaginary intersecting lines. The determinant $\varDelta$ is called the DISCRIMINANT of the equation. We may call $\tau = b_2 - ac = \begin{vmatrix} b & a \\ c & b \end{vmatrix}$ the characteristic determinant, or simply CHARACTERISTIC.

We shall now show that *the discriminant and characteristic of an equation of the second order are not changed by a translation.* By the translation in which x and y are replaced by $x + a_1$ and $y + b_1$, equation (2) is transformed into

$$(14) \quad ax^2 + 2bxy + cy^2 + 2(aa_1 + bb_1 + d)x + 2(ba_1 + cb_1 + e)y \\ + a_1(aa_1 + bb_1 + d) + b_1(ba_1 + cb_1 + e) + da_1 + eb_1 + f = 0.$$

From this it is apparent that τ has remained invariant. The discriminant

$$\varDelta = \begin{vmatrix} a & b & aa_1 + bb_1 + d \\ b & c & ba_1 + cb_1 + e \\ aa_1 + bb_1 + d & ba_1 + cb_1 + e & a_1(aa_1 + bb_1 + d) + b_1(ba_1 + cb_1 + e) \\ & & + da_1 + eb_1 + f \end{vmatrix} =$$

$$\begin{vmatrix} a & b & d \\ b & c & e \\ aa_1 + bb_1 + d & ba_1 + cb_1 + e & da_1 + eb_1 + f \end{vmatrix} =$$

$$\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix},$$

which is the original discriminant.

In a translation τ and Δ are invariants. The same is true of τ and Δ in a rotation, and consequently in any motion of the plane in itself. No special proof for the case of rotation will be given, since it is contained in the linear transformation which will now be treated.

5. A linear transformation (§ 19, XII)

$$(15)\qquad \begin{cases} x=\alpha x'+\beta y'+\xi, \\ y=\gamma x'+\delta y'+\eta, \end{cases}$$

leaves the line at infinity invariant. It may therefore be expected that such a transformation does not materially change the expressions τ and Δ.

As the constants ξ and η mean simply a translation in addition to the special linear transformation

$$(16)\qquad \begin{cases} x=\alpha x'+\beta y', \\ y=\gamma x'+\delta y', \end{cases}$$

and as a translation does not change τ and Δ, it is sufficient to study the effect of (16) upon the general equation (2). Making in (2) for x and y the substitution (16) and afterwards replacing x' and y' by x and y, we have

$$(17)\qquad \begin{aligned}&(a\alpha^2+2b\alpha\gamma+c\gamma^2)x^2+2\{a\alpha\beta+b(\alpha\delta+\beta\gamma)+c\gamma\delta\}xy \\ &\quad +(a\beta^2+2b\beta\delta+c\delta^2)y^2+2(d\alpha+e\gamma)x+2(d\beta+e\delta)y+f=0.\end{aligned}$$

The discriminant of this equation is

$$\Delta'=\begin{vmatrix} a\alpha^2+2b\alpha\gamma+c\gamma^2 & a\alpha\beta+b(\alpha\delta+\beta\gamma)+c\gamma\delta & d\alpha+e\gamma \\ a\alpha\beta+b(\alpha\delta+\beta\gamma)+c\gamma\delta & a\beta^2+2b\beta\delta+c\delta^2 & d\beta+e\delta \\ d\alpha+e\gamma & d\beta+e\delta & f \end{vmatrix},$$

which reduces down to

$$(18)\qquad \Delta'=(\alpha\delta-\beta\gamma)^2\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}=(\alpha\delta-\beta\gamma)^2\Delta.$$

In a similar manner there is found

$$\tau' = (\alpha\delta - \beta\gamma)^2\tau. \tag{19}$$

From this it follows that *the character of a curve of the second order is not changed by a linear transformation.*

It is always assumed that $\alpha\delta - \beta\gamma \neq 0$. Indeed, according as $\tau \gtreqless 0$, also $\tau' \gtreqless 0$; and also as $\varDelta \gtreqless 0$, $\varDelta' \gtreqless 0$.

For a rotation, $\tau' = (\cos^2\theta + \sin^2\theta)\tau = \tau$,

$$\varDelta' = \varDelta;$$

i.e., in this case τ and $\varDelta$ are invariants. In the next section it will be seen that conics are characterized by their pole- and polar involutions on the line at infinity (involution of diameters).

Thus, it is also geometrically evident that in a linear transformation, which does not change the involution of the infinitely distant line, the type of a conic is not changed.

Ex. 1. Find (18) from the preceding unsolved determinant. Calculate also τ'.

Ex. 2. If $b^2 - ac = 0$, assume $b = -\sqrt{ac}$ and transform, with this condition, the general equation (2) to the normal form $y^2 = 2px$.

Ex. 3. Discuss the curve determined by $y^2 - 2xy + x^2 - 1 = 0$.

Ex. 4. What curve does the equation

$$ax^2 + (a+b)xy + by^2 + (a+c)x + (b+c)y + c = 0$$

represent?

Ex. 5. If in the linear transformation $\alpha\delta - \beta\gamma = 0$, i.e., $\dfrac{\alpha}{\beta} = \dfrac{\gamma}{\delta}$, we have

$$x = \beta\left(\frac{\alpha}{\beta}x' + y'\right),$$

$$y = \delta\left(\frac{\gamma}{\delta}x' + y'\right).$$

Hence, no matter what the values of x' and y' may be, $\frac{x}{y}=\frac{\beta}{\delta}=$ constant. The whole $x'y'$-plane is transformed into the straight line $x\delta-y\beta=0$.

§ 32. Polar Involution of Conics. Center. Diameters. Axes. Asymptotes.

1. In §§ 12 and 13 the involutoric properties of the circle have been explained. As a collineation does not change projective properties, it is clear that the following theorems hold for conics. (As the figures of involution referred to in this section are in close analogy with those of the circle, their reproduction is left to the student.)

I. *The polars of the points of a straight line l pass through a fixed point L, the pole of l.*

II. *The poles of the rays of a pencil P lie on a straight line p, the polar of P.*

From this follows immediately

III. *If the pole L of a straight line l lies on a second line g, then the pole G of g lies on l.*

IV. *If the polar l of a point L passes through the pole G of a second line g, then the line g passes through the pole of l.*

Consider now any straight line l and its pole L. Let a be any ray through L cutting l at A. The pole A_1 of a lies on l. Hence the pole of the ray a_1, passing through L and A_1, coincides with A. Taking any number of rays $a, b, c, d, \ldots$ and constructing as before their corresponding rays $a_1, b_1, c_1, d_1, \ldots$, a system of coincident polars and poles through L and on l are obtained which are in involution; i.e., every pair of corresponding rays and poles, xx_1 and XX_1, are harmonic with the double-rays and double-points through L and on l, respectively. The double-elements are real when l intersects the conic really; i.e., when L admits of two real tangents to the conic. If l does not intersect in real points, then the involution has imaginary double-elements. In case that l is tangent to the conic, the double-

elements are coincident. The points of intersection of l and the tangents from L to the conic, whether real, coincident, or imaginary, give in all cases the double-elements of the involution. Accordingly, *hyperbolic*, *parabolic*, and *elliptic* involutions are distinguished.

Two corresponding rays x, x_1, and l, with their poles X, X_1, and L always form a self-polar triangle; i.e., a triangle whose vertices are the poles of its opposite sides.

All these properties might be derived directly from the generation of conics by projective pencils and ranges; i.e., without reference to the perspective of the circle. We have used this method to lay particular stress upon the invariance of these properties by projective transformations. Examples will be given later on to show how some of the propositions (all, for that matter) in this connection may be derived independently of perspective.

2. It is now of the greatest interest to investigate the involutions of poles and polars when the latter are assumed in special positions. Let l be at an infinite distance. Then for every ray a through L cutting the conic at P and P_1 $(LAPP_1) = -1 = (L \infty PP_1)$, or $LP = -LP_1$. Every ray through L, the pole of the line at infinity, therefore cuts the conic in two points which are equally distant from L. This point is therefore called the CENTER of the conic.

To every ray through the center corresponds an infinitely distant pole. Call the center O. Of great importance is the polar involution through O. The poles A and A_1 of any two corresponding rays through O are infinitely distant. Any ray through A cutting a_1 in B and the conic in C and D is parallel to a, and $(BACD) = (B \infty CD) = -1$; hence $BC = -BD$; i.e., B bisects CD. Two corresponding rays of the involution through O are called CONJUGATE DIAMETERS of the conic.

Including imaginary elements, the foregoing properties of the polar involution at the center give the following theorems:

V. *All chords of a conic parallel to a diameter are bisected by its conjugate diameter.* The relation between conjugate diameters is reversible.

VI. *If a diameter intersects a conic, then the tangents at the points of intersection are parallel to the conjugate diameter.*

3. DEFINITIONS.—The rectangular pair of the polar involution at the center are called the AXES of the conic.

The double-rays of the involution of diameters of a conic are called the ASYMPTOTES.

The points at which the polar involution is rectangular are called the FOCI of the conic.

In these definitions it is assumed that the involutions exist. From the definition of a self-polar triangle it is easily concluded that the involutions on two of its sides are hyperbolic, while on the third it is elliptic. If we now consider a self-polar triangle having the center O of the conic as one of its vertices, then two distinct cases may occur.

First. The involutions of poles on two conjugate diameters may both be hyperbolic, while the polar involution at the center, consequently the involution of poles on the line at infinity, is elliptic. *The involution of conjugate diameters has no real double-rays; the conic is an ellipse.*

Second. The involution at the center is hyperbolic; it is hyperbolic on one diameter and elliptic on the conjugate diameter. *The involution of conjugate diameters has real double-rays; i.e., the conic has real asymptotes and is an hyperbola.*

From theorem V it follows immediately that *the ellipse and hyperbola are symmetrical with respect to both axes.*

The existence of the ellipse, hyperbola, and a special involution of diameters in connection with a conic, called *parabola*, will be proved in the next section. At the same time the existence of foci will be proved.

§ 33. Existence of Ellipse, Hyperbola, Parabola, and their Foci.

1. In Fig. 40, just as in Fig. 19, § 15, let s be the axis, q' and r the counter-axes, and C the center of collineation. Assume any circle K with C as a center and determine the pole O of r with

respect to K. Draw also the polar involution at O, of which OM and OM_1 are a corresponding pair intersecting K at X, Y and X_1, Y_1, respectively. Now $(OMXY)=(OM_1X_1Y_1)=-1$. In the central projection r and consequently M and M_1 are projected

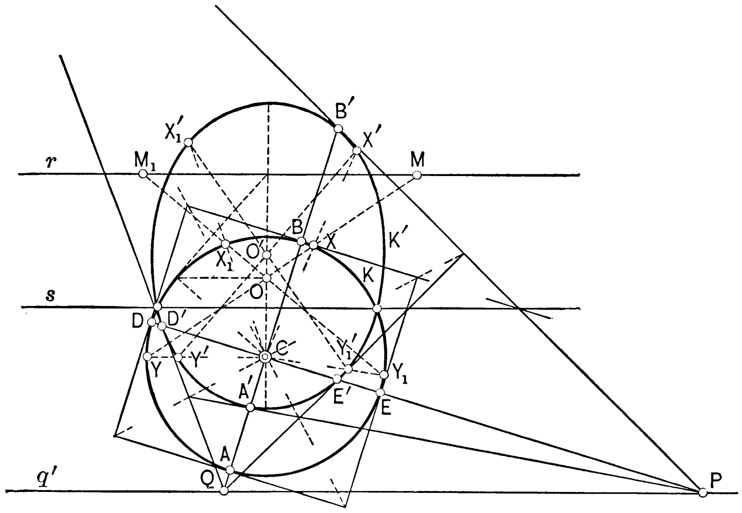


FIG. 40.

to infinity. Hence, designating the projected figure by primes, $(O'\infty X'Y')=(O'\infty X_1'Y_1')=-1$; i.e., $O'X'=-O'Y'$, $O'X_1'=-O'Y_1'$; O' is the center of the transformed circle, and the polar involution at O' becomes the involution of diameters. Now the polar involution at O is elliptic, parabolic, or hyperbolic according as r does not cut K, touches K, or intersects it in two points. The same is evidently true of the involution of diameters at O'. In case that r is tangent to K, O coincides with the point of tangency of r; O' is projected to infinity, so that the diameters become all parallel. This is the case of the parabola. A parabola may therefore be considered as a conic tangent to the line at infinity. With the existence of these different involutions of diameters the existence of the ellipse, the parabola, and the hyperbola is proved. To sum up, *the central projection of a circle is an*

ellipse, a parabola, or an hyperbola according as it does not intersect r*, is tangent to* r*, intersects* r. Further, *a conic is an ellipse, a circle, a parabola, or an hyperbola according as the involution of diameters is elliptic, elliptic and rectangular, parabolic (parallel diameters with infinite center), or hyperbolic.*

2. Consider now the rectangular polar involution at the center O of the circle K. The central projection of O coincides with itself, and for the corresponding pair of two perpendicular diameters AB and DE, $A'B' \perp D'E'$. The points of intersection P and Q of DE and AB with q' are the poles of $A'B'$ and $D'E'$. The vanishing line q' is therefore the polar of C with regard to K'. The same holds for any pair of perpendicular diameters of K and their transformations.

The involution of polars at C *of* K' *is therefore rectangular;* C *is a focus of* K'.

Any conic which is the perspective of a circle with the center of perspective as a center has this center as a focus.

The construction also shows that a focus lies on the major axis of the conic.

Ellipse and hyperbola are symmetrical with respect to their axes; both curves have therefore two foci (real). That a conic cannot have more than two real foci is seen from the construction of Fig. 40, and also from the fact that every point not on the axes admits of oblique pairs of polars.

The double-rays of the rectangular polar involutions at the foci pass through the circular points at infinity; they may be considered as imaginary tangents to the conic from its foci. The foci of a conic may therefore also be defined as follows:

The foci of a conic are the points of intersection of the tangents from the circular points at infinity to the conic.

Two of these intersection-points are real, the other two are imaginary and, on account of the symmetry, are necessarily situated on the other axis. Ellipse and hyperbola admit, therefore, also of two imaginary foci.

Analytically this also appears by writing the equations of

the imaginary double-rays at the foci, when their distance from the center of the conic is c:

$$\begin{aligned} &1. \quad (x-c)+iy=0. \\ &2. \quad (x-c)-iy=0. \\ &3. \quad (x+c)+iy=0. \\ &4. \quad (x+c)-iy=0. \end{aligned}$$

From 1 and 2, $x=c;\ y=0.$
From 3 and 4, $x=-c;\ y=0.$
From 1 and 4, $x=0;\ y=-ic.$
From 2 and 3, $x=0;\ y=+ic.$

The solutions of 1 and 3, 2 and 4 give the circular points.

§ 34. Construction of Foci Independent of Central Projection.

To the pencil T of parallel rays $a, b, c, \ldots$ considered as polars of the conic K, Fig. 41, correspond the poles $A', B', C', \ldots$ on the conjugate diameter n of the direction of T. To the diameter $m \parallel T$ corresponds as pole the infinitely distant point of n. To the line at infinity belonging to the pencil T corresponds as pole the center I' of K.

From $A', B', C', \ldots, I'$, draw rays $a', b', c', \ldots, j'$ perpendicular to m. These rays form another pencil S of parallel rays which is projective to the pencil T. As these two pencils are perpendicular, their intersection $A_1, B_1, C_1, \ldots, M_1$ is an equilateral hyperbola, having m and its perpendicular through I' as asymptotes. Both pencils T and S intersect the axes each in two coincident projective ranges, for instance

$$(ABC \ldots) \barwedge (A_2B_2C_2 \ldots).$$

On this axis, to the point I' of the first range corresponds the infinitely distant one on the same axis. If I' is taken as a point of the second range, then its corresponding one is infinitely distant. Hence I' and the point at infinity on the horizontal

axis may be interchanged without disturbing the projectivity; the foregoing point-ranges form therefore an involution (§ 3). The double-points of this involution are the points where the equilateral hyperbola H cuts the axis. To the ray j (pencil T) through one of these points, say F, corresponds the perpendicular ray j' in the pencil S. But every ray of S passes through the

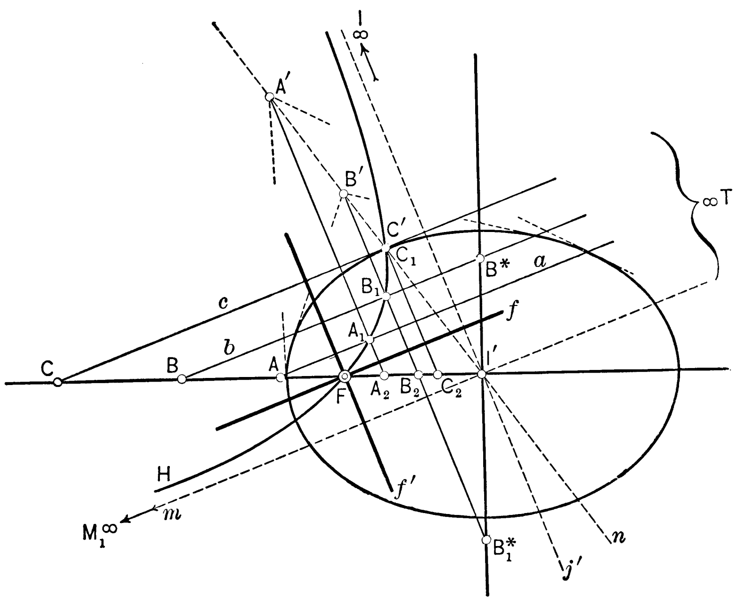


FIG. 41.

pole of the corresponding ray in T. Hence j' contains the pole of j, and j and j' form consequently a rectangular pair of the polar involution around F. The polar involution of any point on an axis contains, however, another rectangular pair, namely, the axis itself and the perpendicular to it through the given point. The polar involution about F has therefore two rectangular pairs and is consequently itself rectangular (§ 5). The point F, according to definition, is therefore a focus. As there are two double-points of the involution $(ABC\ldots)\,\overline{\wedge}\,(A_2B_2C_2\ldots)$,

there are also two foci. Assuming that the points of a pair AA_2 are on the same side of the center of involution, then

$$I'A \cdot I'A_2 = +k^2,$$

and $I'F = k$; the foci are real. But on the other axis $I'A^* \cdot I'A_2^* = -k^2$, i.e., the double-points of the involution, or the foci, are imaginary. This is in accordance with the statement in the foregoing section.[1]

Ex. 1. Carry out construction of this section on a large sheet.

Ex. 2. Instead of taking an ellipse for the conic K, take an hyperbola.

§ 35. Focal Properties of Conics.

1. From Fig. 40, § 33, we have, if R designates the point of intersection of PD with r (not shown in the figure),

$$(C\infty DR) = (CPD'\infty),$$

or

$$CD : CR = CD' : PD'.$$

Designating the distances of C and D' from r and q' by γ and δ, respectively, there is

$$CR : PD' = \gamma : \delta = CD : CD';$$

consequently

$$\frac{CD'}{\delta} = \frac{CD}{\gamma} = \text{constant}.$$

This result may be stated by the theorem:

The ratio between the distance of any point of a conic from one of its foci and the distance of the point from the polar of this focus is constant.

---

[1] Since an involution of right angles does not admit of real double-rays, it follows that the foci are within the conic; i.e., within that portion of the plane from which no real tangents may be drawn. They are situated on an axis, since in any other case the polar involution would have oblique pairs.

DEFINITION.—*The polar of a focus of a conic is called directrix.*

This theorem, as well as its converse, may be used to define conics, as was done by PAPPUS (Mathematical Collections):

The locus of a point whose distances from a fixed point and a fixed straight line (*not passing through the fixed point*) *have a constant ratio is a conic.* The fixed point is the focus, the fixed line the corresponding directrix.

If, in Fig. 40, K does not intersect r, i.e., if $\dfrac{CD}{r} < 1$, K' is an ellipse; if K touches r, $\dfrac{CD}{r} = 1$, and K' is a parabola; if K intersects r, $\dfrac{CD}{r} > 1$, and K' is an hyperbola. The figure may easily be drawn for the case of an hyperbola or a parabola.

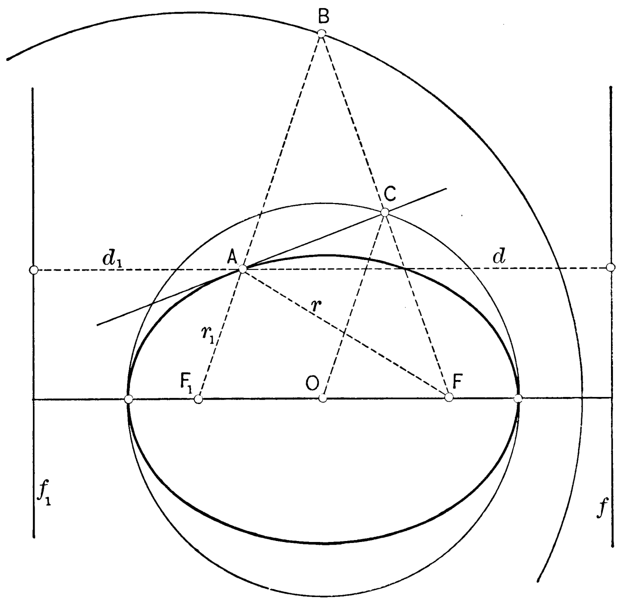


FIG. 42.

2. From Pappus' metrical definition a number of properties of conics may be derived. Given a conic K and its foci F and F_1, Figs. 42 and 43, according as we take for K an ellipse or an hy-

perbola. Both curves are symmetrical with respect to both axes. The ratios $\left(\frac{CD}{r}\right)$ are therefore the same for both foci and their corresponding polars (directrices). Taking any point A on K and designating the focal distances AF and AF_1 by r and r_1, and the distances from the corresponding directrices by d and d_1, we have

$$\frac{r}{d}=\frac{r_1}{d_1}=\text{constant (Pappus).}$$

From this $\frac{r\pm r_1}{d\pm d_1}$=same constant as above. But in an ellipse

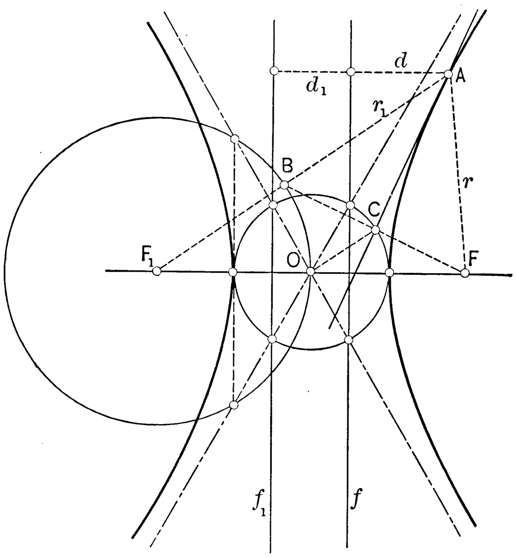


FIG. 43.

$d+d_1$=constant, and in an hyperbola $d-d_1$=constant. Hence the theorem:

The sum of the radii vectores (AF, AF_1) of any point of an ellipse is constant.

The difference of the radii vectores of an hyperbola is constant. In both cases the constant is equal to the distances of the vertices of the curves.

The second part of this theorem results by taking A in one of the vertices. In case of an ellipse we have then

$$\frac{r+r_1}{d+d_1}=\frac{r}{d}, \quad r+r_1=r+\frac{r}{d}d_1;$$

in case of an hyperbola

$$r-r_1=r-\frac{r}{d}d_1.$$

3. Given a conic K, Fig. 44, and its foci F and F_1. Take any point P in the plane of K and construct the polar involution around P and its rectangular pair PR_1, PR. Connecting P with all pairs of the involution on the axis which are formed by couples of rectangular conjugate polars of K parallel to PR

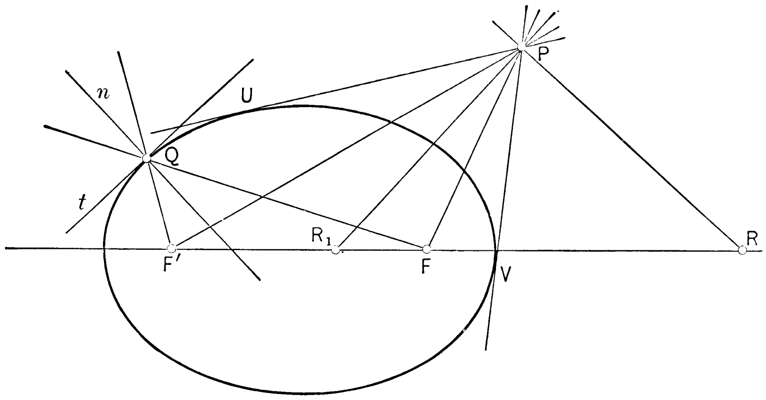


FIG. 44.

and PR_1 (see Fig. 41, § 34), an involution of rays at P is obtained in which PF and PF' are the double-rays, and PR, PR_1 the rectangular pair. PR and PR_1 are consequently the bisectors of the angles formed by PF and PF_1. In the polar involution around P, the tangents PU and PV from P to K are the double-rays and, according to the construction, PR, PR_1 the rectangular pair. The angles formed by PU and PV are therefore also bisected by PR and PR_1. Hence the theorem:

The angles formed by the tangents (real) from a point to a conic are bisected by the bisectors of the rays joining this point to the foci; or these tangents form equal angles with the focal rays (PF, PF_1).

If P lies on K, say at Q, then PU and PV coincide with the tangent t at Q. We have therefore the corollary:

The tangent at any point of a conic includes equal angles with the focal radii at this point.

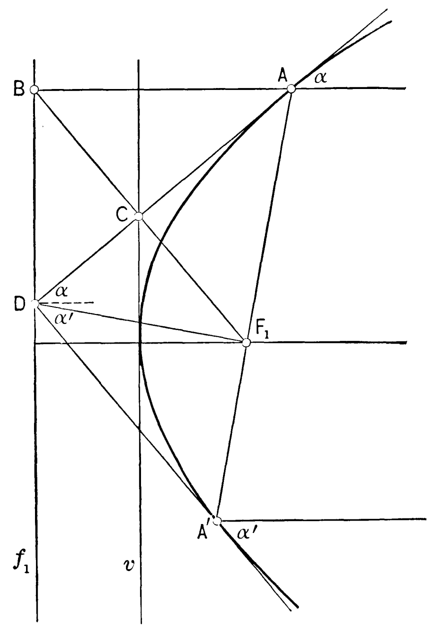


FIG. 45.

4. If in Figs. 42 and 43 F_1A is produced and AB made equal to AF ($F_1B=2a$, the major axis), then $BF \perp AC$ (tangent at A); hence $BC=FC$. As $OC \parallel F_1B = \frac{1}{2}F_1B$, we have the theorems:

The locus of the reflected images of a focus on all tangents of an ellipse or an hyperbola is a circle having the other focus as a center and the major axis as a radius.

The locus of the foot-points of all perpendiculars from the foci

of an ellipse, or an hyperbola, to their tangents is a circle over the major axis as a diameter.

In case of a parabola, Fig. 45, the first circle becomes the directrix f_1 and the second the tangent v at the vertex. Suppose D to be a point where two perpendicular tangents of the parabola meet, and let A and A' be the points of tangency. We have $\alpha+\alpha'=\frac{\pi}{2}$; hence AF_1 and $A'F_1$ include an angle of π. D is therefore the pole of a focal chord AA' and, as such, lies in the directrix. Hence:

The tangents from any point of the directrix of a parabola to the parabola are perpendicular to each other.

It is known from § 33 that the polar involutions around a focus are rectangular. Thus, if AA' is a focal chord and A_1 its pole on the directrix, $FA_1 \perp AA'$ at F. The foregoing statement is therefore only a part of the general proposition, since $\angle AFA_1=$ right angle, Fig. 46:

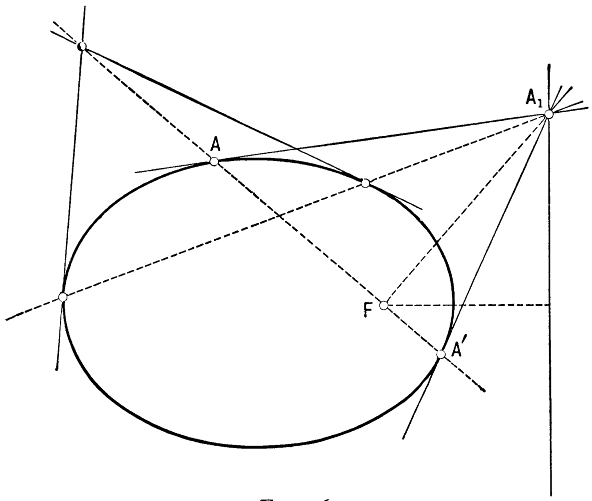


FIG. 46.

The portion of a tangent of a conic between its point of tangency and its intersection with the corresponding directrix appears as subtended by a right angle when seen from the focus.

Further, from the fact that the polar of a point which is situated on another polar passes through the pole of this polar, and that the polar involution at a focus is rectangular it follows, Fig. 46:

The rays joining a focus with the point of intersection of two tangents and the point of intersection of the chord of contact of these tangents with the corresponding directrix are perpendicular.

Ex. 1. If from any foint O at a distance c from the center of a circle with radius r two perpendicular secants are drawn, intersecting the circle in the points A, B, C, D, then

$$OA^2+OB^2+OC^2+OD^2=4r^2.$$

Assuming this proposition, prove:

The locus of the point of intersection of two tangents to an ellipse or an hyperbola which cut at right angles is a concentric circle. If a and b are the parameters (major and minor half-axes), the radius of the circle is $\sqrt{a^2 \pm b^2}$.

Ex. 2. Let Q and R be the points of intersection of a third tangent with the two tangents to a parabola from a point P. Prove that $\angle RPF$ is equal to the angle which QP makes with the diameter of the parabola through P.

Ex. 3. Applying the proposition established in the foregoing exercise, prove that *the circle circumscribing a triangle formed by three tangents to a parabola passes through the focus.*

Ex. 4. *The locus of the foci of all parabolas which touch the three sides of a given triangle is the circumscribing circle of the triangle.* (Cremona.)

Ex. 5. Suppose a quadrilateral $ABCD$ is circumscribed about a conic with the points of tangency $KLMN$, Fig. 47. The pairs of sides AB and CD, BC and AD, CA and BD intersect each other in the three points OPQ. The pole of AC is the point of intersection of KN and LM and is the point of concurrence of PO, ML, DB, NK. Similarly, KL, AC, NM meet at a point of PO, the pole of BD. Hence, in a quadrilateral circumscribed to a conic, the diagonals form a self-polar triangle. If A, B, N, and L are given, then we may choose Q

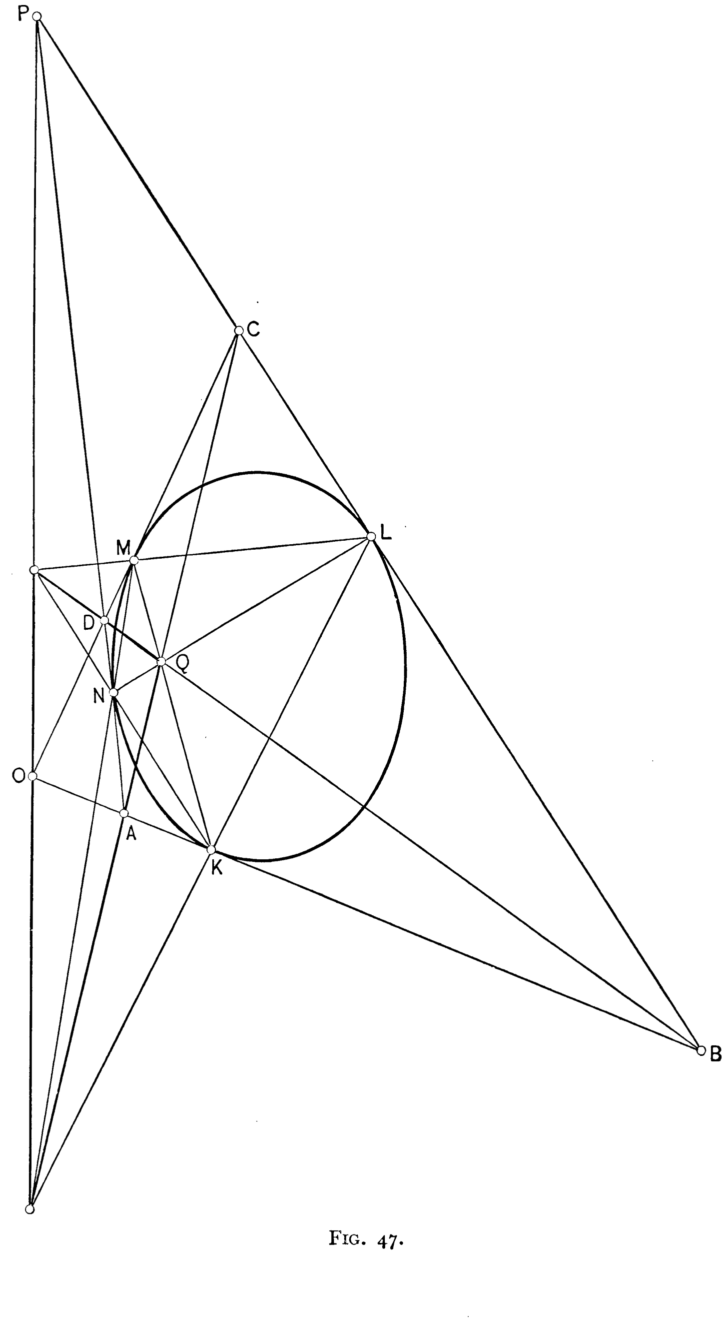


FIG. 47.

at random on NL, by which the fourth side CD and its point of contact is perfectly determined. From this the proposition follows:

Let PAB be a triangle circumscribed to a conic, and LN the chord of contact of the tangents PB and PA; then the lines joining any point Q on LN to A and B are conjugate polars.

Ex. 6. Prove: *The portion of a movable tangent of a central conic between the two tangents at the vertices subtends right angles at the foci.*

Ex. 7. *The lines joining the points of intersection of all circles through the foci with the tangents at the vertices of a central conic are the tangents of this conic.* (The two points joined must not be equally distant from the axis.)

Ex. 8. Consider again, Fig. 48, two tangents intersecting at P, and let their chord of contact AB intersect the directrix

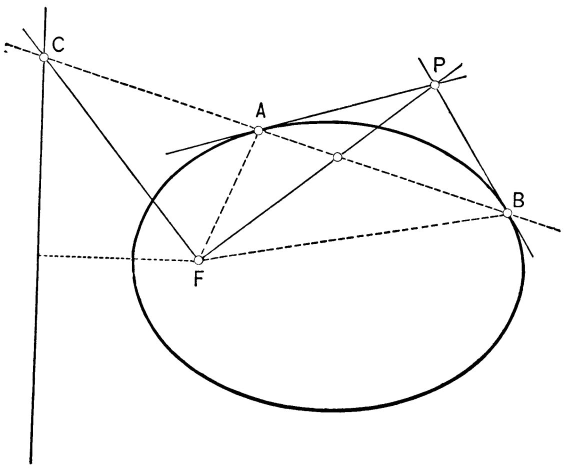


Fig. 48.

at C. Then PF is the polar of C. Consequently $F \cdot APBC$ is a harmonic pencil in which one pair, FP, FC is perpendicular. FP therefore bisects the angle AFB, § 5. Hence the proposition:

The line joining a focus to the point of intersection of two tan-

gents of a conic bisects the angles formed by the rays joining the focus to the points of contact.

Ex. 9. Consider two fixed tangents and a movable tangent of a conic, and join a focus to their points of intersection and points of contact. Applying the proposition of Ex. 8, prove that *the piece of a movable tangent included by two fixed tangents appears under a constant angle from a focus.*

In case of an hyperbola whose asymptotes include an angle ϕ, the above constant angle with reference to the asymptotes as fixed tangents is $\pi-\frac{\phi}{2}$.

Ex. 10. *The extremities of a tangent, determined by the asymptotes, and the foci of the hyperbola are concyclic.*

By means of this proposition it is easy to construct an hyperbola by its tangents when the asymptotes and the foci are known.

§ 36. Analytical Expression for Tangent and Polar.

1. Although problems connected with tangents and polars of general conics have so far been simply treated without their analytic forms, it is of great value for the developments that will follow to establish their equations by means of transversals and anharmonic ratios. Let (x_1, y_1), (x_2, y_2) be two points A and C; then the coordinates of any point B of the straight line AC are given by the equations

$$(1) \qquad x=\frac{x_1-\lambda x_2}{1-\lambda}, \quad y=\frac{y_1-\lambda y_2}{1-\lambda}, \quad \text{where} \quad \lambda=\frac{AB}{CB}.$$

Substituting these values in the general equation

$$u=ax^2+2bxy+cy^2+2dx+2ey+f=0,$$

and multiplying by $(1-\lambda)^2$, we obtain, after arranging according to ascending powers of λ,

(2) $$u_1 - 2\lambda v + \lambda^2 u_2 = 0,$$

where

$$u_1 = ax_1^2 + 2bx_1y_1 + cy_1^2 + 2dx_1 + 2ey_1 + f,$$

$$u_2 = ax_2^2 + 2bx_2y_2 + cy_2^2 + 2dx_2 + 2ey_2 + f,$$

$$v = x_1(ax_2 + by_2 + d) + y_1(bx_2 + cy_2 + e) + dx_2 + ey_2 + f$$

$$= x_2(ax_1 + by_1 + d) + y_2(bx_1 + cy_1 + e) + dx_1 + ey_1 + f.$$

From (2) two values of λ are obtained which when substituted in (1) give the coordinates of the points of intersection of AC with the conic U. In case that these two points coincide, the roots of (2) will be equal and AC is a tangent to U. Now, the condition for equal roots is

(3) $$v^2 - u_1u_2 = 0.$$

Suppose that (x_2, y_2) itself is on U, then $u_2 = 0$, and the condition reduces to $v = 0$. Every point (x_1, y_1) which satisfies $v = 0$ lies on the tangent at (x_2, y_2). The equation of the tangent at (x_2, y_2) is therefore

(4) $$x(ax_2 + by_2 + d) + y(bx_2 + cy_2 + e) + dx_2 + ey_2 + f = 0.$$

2. Let A and B be the points (x_1, y_1), (x_2, y_2) and C, D the points of intersection of the straight line AB with U, corresponding to the roots of (2). If A, B and C, D are two harmonic pairs, then $(ABCD) = -1$; i.e., $\frac{AC}{BC} : \frac{AD}{BD} = -1$, or $\frac{AC}{BC} + \frac{AD}{BD} = 0$.

But $\frac{AC}{BC}$ and $\frac{AD}{BD}$ are the roots of (2); hence $ABCD$ form a harmonic group if the sum of the roots of (2); i.e., $2\frac{v}{u_2} = 0$. As u_2 is not supposed to be on U, this is only possible if $v = 0$; i.e., if

(5) $$x_2(ax_1 + by_1 + d) + y_2(bx_1 + cy_1 + e) + dx_1 + ey_1 + f = 0.$$

If four points $ABCD$, of which C and D are on U, are collinear and form a harmonic group, then the coordinates of A and B are related by (5). Keeping the point A fixed and letting B under condition (5) vary, it is plain that all points B restrained by these conditions lie on the straight line

$$(6) \qquad x(ax_1+by_1+d)+y(bx_1+cy_1+e)+dx_1+ey_1+f=0.$$

This line is called the polar of the point A (x_1, y_1) with respect to U.

Similarly, the polar of $B(x_2, y_2)$ is given by

$$(7) \qquad x(ax_2+by_2+d)+y(bx_2+cy_2+e)+dx_2+ey_2+f=0.$$

If (x_2, y_2) lies on the polar (6) of (x_1, y_1), then (5) holds. But this can also be written as

$$(8) \qquad x_1(ax_2+by_2+d)+y_1(bx_2+cy_2+e)+dx_2+ey_2+f=0,$$

which is the condition that (x_1, y_1) lies on the polar of (x_2, y_2). Hence the theorem which has already been established before:

If A is on the polar of B, then B is on the polar of A.

From this it follows that the polars of the points of a straight line all pass through its pole, and, conversely, the poles of all rays through a fixed point lie on its polar.

3. To establish the relation between the points of a straight line and the corresponding pencil of polars, assume first any four lines through a fixed point:

$$\begin{aligned} p_1 &\equiv a_1x+b_1y+c_1=0, \\ p_2 &\equiv a_2x+b_2y+c_2=0, \\ p_1-\lambda p_2&=0, \\ p_1-\mu p_2&=0. \end{aligned}$$

Cut these lines by any transversal and find the anharmonic ratio of the four points of intersection $A_1A_2A_3A_4$. For the sake of simplicity, choose the x-axis as this transversal, so that the distances of these points from the origin become

$$-\frac{c_1}{b_1}, \quad -\frac{c_2}{b_2}, \quad -\frac{c_1-\lambda c_2}{b_1-\lambda b_2}, \quad -\frac{c_1-\mu c_2}{b_1-\mu b_2}.$$

The anharmonic ratio is easily found:

$$(9) \qquad (A_1A_2A_3A_4) = \frac{A_1A_3}{A_1A_4} : \frac{A_2A_3}{A_2A_4} = \frac{\lambda}{\mu}.$$

If now (x_1, y_1), (x_2, y_2) are the coordinates of two points A_1, A_2, then the coordinates of any point A_3 on A_1A_2 are

$$\frac{x_1 - \lambda x_2}{1-\lambda}, \quad \frac{y_1 - \lambda y_2}{1-\lambda}, \quad \text{where} \quad \lambda = \frac{A_3A_1}{A_3A_2}.$$

The polar of A_3 is

$$\begin{aligned} &x\left(a \cdot \frac{x_1 - \lambda x_2}{1-\lambda} + b \cdot \frac{y_1 - \lambda y_2}{1-\lambda} + d\right) \\ +\,&y\left(b \cdot \frac{x_1 - \lambda x_2}{1-\lambda} + c \cdot \frac{y_1 - \lambda y_2}{1-\lambda} + e\right) \\ +\,&\quad d \cdot \frac{x_1 - \lambda x_2}{1-\lambda} + e \cdot \frac{y_1 - \lambda y_2}{1-\lambda} + f = 0, \end{aligned}$$

or, multiplying by $1-\lambda$ and rearranging,

$$\begin{aligned} &x(ax_1 + by_1 + d) + y(bx_1 + cy_1 + e) + dx_1 + ey_1 + f \\ &\qquad - \lambda\{x(ax_2 + by_2 + d) + y(bx_2 + cy_2 + e) + dx_2 + ey_2 + f\} = 0. \end{aligned}$$

Designating the equations of the polars of A_1, A_2 simply by $p_1 = 0$, $p_2 = 0$, the polar of A_3 will be represented by $p_1 - \lambda p_2 = 0$.

Analogously, the polar of a fourth point A_4 on A_1A_2 is represented by $p_1 - \mu p_2 = 0$, where $\mu = \dfrac{A_4A_1}{A_4A_2}$. The anharmonic ratio of A_1, A_2, A_3, A_4 is evidently $\dfrac{A_3A_1}{A_3A_2} : \dfrac{A_4A_1}{A_4A_2} = \dfrac{A_1A_3}{A_1A_4} : \dfrac{A_2A_3}{A_2A_4} = \dfrac{\lambda}{\mu}$. According to (9) the same is true of the four points of intersection of any transversal with the polars $p_1 = 0$, $p_2 = 0$, $p_1 - \lambda p_2 = 0$, $p_1 - \mu p_2 = 0$ of the points A_1, A_2, A_3, A_4. Hence the theorem:

The range of points of a straight line and the corresponding pencil of polars are projective.

This follows also by considering the polar involution around a point and the corresponding involution of poles on its polar, as shown elsewhere.

4. EQUATION OF A CONIC IN LINE-COORDINATES.

To find the equation in line-coordinates u, v of a conic with the Cartesian equation

$$ax^2+2bxy+cy^2+2dx+2ey+f=0, \tag{10}$$

consider the equation of a tangent

$$(ax_1+by_1+d)x+(bx_1+cy_1+e)y+(dx_1+ey_1+f)=0 \tag{11}$$

at a point (x_1, y_1) of this conic. The line coordinates of this tangent are

$$\left\{\begin{aligned} u&=\frac{ax_1+by_1+d}{dx_1+ey_1+f}, \\ v&=\frac{bx_1+cy_1+e}{dx_1+ey_1+f}. \end{aligned}\right. \tag{12}$$

Conversely, the Cartesian coordinates x_1, y_1 expressed in terms of the line-coordinates u and v of the tangent at this point are, from (12),

$$\left\{\begin{aligned} x_1&=\frac{(cf-e^2)u+(de-bf)v+(be-cd)}{(be-cd)u+(bd-ae)v+(ac-b^2)}, \\ y_1&=\frac{(de-bf)u+(af-d^2)v+(bd-ae)}{(be-cd)u+(bd-ae)v+(ac-b^2)}. \end{aligned}\right. \tag{13}$$

But these values of x_1 and y_1 satisfy (10). Thus, substituting (13) in (10), we get the relation which exists between the line-coordinates u and v of the tangents of the conic (10), or the equation of the conic in line-coordinates. After reduction this equation becomes

$$\begin{aligned}(cf-e^2)u^2+2(de-bf)uv+(af-d^2)v^2+& \\ 2(be-cd)u+2(bd-ae)v+ac-b^2&=0.\end{aligned} \tag{14}$$

Ex. 1. Find the line-equations of

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1;$$

$$y^2 = px;$$

$$x^2 + y^2 = r^2;$$

$$(x-a)^2 + (y-b)^2 - r^2 = 0.$$

Ex. 2. Establish the equation of a point of (14).

Ex. 3. From the line-equation of a conic,

$$au^2 + 2buv + cv^2 + 2du + 2ev + f = 0,$$

establish the Cartesian equation.

§ 37. Theory of Reciprocal Polars.

1. We have already discussed the principle of duality, § 22, in an elementary manner. In this section it will be seen that the principle follows directly from the theory of polars.

To every point as a pole corresponds a straight line as a polar, and conversely. To two projective pencils producing a conic correspond, according to the theorem at the end of § 36, 3, two projective point-ranges which produce a conic as an envelope; i.e., to the points of a conic correspond the tangents of another conic, called the reciprocal of the first. In general, to any figure consisting of points and straight lines corresponds a figure consisting of straight lines and points, the polars and poles of the points and lines of the first figure. The anharmonic ratios of corresponding elements are the same in the original and reciprocal figure.

The transformation thus established is called *polar reciprocity* and is contained in the slightly more general principle of duality. The polar of a point (x_1, y_1) with respect to the conic U is given by

$$(1) \qquad x(ax_1 + by_1 + d) + y(bx_1 + cy_1 + e) + dx_1 + ey_1 + f = 0,$$

or, introducing the line-coordinates,

$$(2)\qquad \begin{cases} u=\dfrac{ax_1+by_1+d}{dx_1+ey_1+f}, \\ v=\dfrac{bx_1+cy_1+e}{dx_1+ey_1+f}, \end{cases}$$

$$(3)\qquad xu+yv+1=0.$$

Formulas (2) are the analytical expression for this transformation. To the point (x_1, y_1) corresponds the straight line with the coordinates (u, v). As the transformation is involutoric; i.e., that the coincidence of a point and straight line necessitates the coincidence of their polar and pole, we can interchange (x, y) with (x_1, y_1), as has been already established.

Designate now the original conic by U, the conic to be reciprocated by K_1 and the reciprocal conic by K_2. Assume U and K_1 as central conics. If the center of U is outside of K_1, two tangents from it may be drawn to K_1, which when reciprocated are two points of K_2. As these tangents pass through the center of U, their poles will be infinitely distant. From this it follows that K_2 is an hyperbola. If K_1 passes through the center of U, then only one real tangent can be drawn to K_1 at this point; i.e., K_2 will have only one infinite point (tangent) and is therefore a parabola. When the center of U is inside of K_1, no real tangents from it can be drawn to K_1; i.e., K_2 has no infinite points and is consequently an ellipse. Hence the theorem:

According as the center of the original conic U is outside, on, or inside of the conic K_1 to be reciprocated, the reciprocal conic K_2 will be an hyperbola, a parabola, or an ellipse.

2. According to (2) (x_1, y_1) is the pole of the line with the coordinates (u, v). Suppose now that this line envelopes a circle of radius r and having its center in the origin of coordinates. The line-equation of this circle is

$$(4)\qquad u^2+v^2=\frac{1}{r_2}.$$

Taking $U=(x-\alpha)^2+(y-\beta)^2-\rho^2=0$; i.e., $a=1$, $b=0$, $c=1$, $d=-\alpha$, $e=-\beta$, $f=\alpha^2+\beta^2-\rho^2$, substituting these values in (2), and finally substituting the values of u and v, thus obtained in (4), we get

$$\left\{\frac{x_1-\alpha}{-\alpha x_1-\beta y_1+\alpha^2+\beta^2-\rho^2}\right\}^2+\left\{\frac{y_1-\beta}{-\alpha x_1-\beta y_1+\alpha^2+\beta^2-\rho^2}\right\}^2=\frac{1}{r^2},$$

or, expanded and rearranged,

$$\begin{aligned}(5)\quad (r^2-\alpha^2)x_1^2-2\alpha\beta x_1y_1+(r^2-\beta^2)y_1^2+2\alpha(\alpha^2+\beta^2-\rho^2-r^2)x_1 \\ +2\beta(\alpha^2+\beta^2-\rho^2-r^2)y_1+r^2(\alpha^2+\beta^2)-(\alpha^2+\beta^2-\rho^2)^2=0.\end{aligned}$$

This is the equation to which the poles of all tangents of (4) are subjected. Hence (5) is the equation of the conic reciprocal to the circle $x^2+y^2=r^2$ with respect to the circle $(x-\alpha)^2+(y-\beta)^2=\rho^2$. Here the characteristic determinant of (5) is

$$\tau=r^2(\alpha^2+\beta^2-r^2).$$

Evidently $\tau>, =, <0$, according as $\alpha^2+\beta^2>, =, >r^2$. Hence, according as the center of U is outside of, on, or inside of (4), the reciprocal conic (5) is an hyperbola, a parabola, or an ellipse, which is in agreement with the previous result.

Ex. 1. What is the reciprocal of a polygon circumscribed to a conic with respect to this conic?

Ex. 2. Find the reciprocal of the point $Au+Bv+C=0$.

Ex. 3. Find the reciprocal of the envelopes $u^2-v^2=0$; $u^2-v^2=1$; $uv=u^2+v^2$.

Ex. 4. Discuss reciprocation in the case where in formulas (2) the determinant

$$\begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}=0.$$

Ex. 5. Given the polar-reciprocal transformation

$$u=\frac{Ax+By+D}{Dx+Ey+F},$$

$$v=\frac{Bx+Cy+E}{Dx+Ey+F},$$

by which to every point (x, y) corresponds a straight line (u, v), and conversely.

Establish the equation of the conic, for which every point (x, y) coincides with the corresponding line (u, v).

§ 38. General Reciprocal Transformation. Polar Systems.

1. Formulas (2) of the foregoing section may be generalized by setting

$$(1)\qquad \begin{cases} u=\dfrac{a_1x+b_1y+c_1}{a_3x+b_3y+c_3}, \\[2ex] v=\dfrac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}. \end{cases}$$

Solving (1) for x and y, we get

$$(2)\qquad \begin{cases} x=\dfrac{(b_2c_3-b_3c_2)u+(b_3c_1-b_1c_3)v+(b_1c_2-b_2c_1)}{(a_2b_3-a_3b_2)u+(a_3b_1-a_1b_3)v+(a_1b_2-a_2b_1)}, \\[2ex] y=\dfrac{(a_3c_2-a_2c_3)u+(a_1c_3-a_3c_1)v+(a_2c_1-a_1c_2)}{(a_2b_3-a_3b_2)u+(a_3b_1-a_1b_3)v+(a_1b_2-a_2b_1)}. \end{cases}$$

To a straight line

$$ax+by+c=0$$

corresponds by this transformation a point with the line-equation

$$(3)\qquad \begin{cases} \{a(b_2c_3-b_3c_2)+b(a_3c_2-a_2c_3)+c(a_2b_3-a_3b_2)\}u+ \\ \{a(b_3c_1-b_1c_3)+b(a_1c_3-a_3c_1)+c(a_3b_1-a_1b_3)\}v+ \\ \{a(b_1c_3-b_2c_1)+b(a_2c_1-a_1c_2)+c(a_1b_2-a_2b_1)\}=0. \end{cases}$$

Conversely, to a point

$$au + bv + c = 0$$

corresponds a line with the equation

$$(4) \qquad (aa_1 + ba_2 + ca_3)x + (ab_1 + bb_2 + cb_3)y + (ac_1 + bc_2 + cc_3) = 0.$$

If four points, of which no three are collinear, with the equations $a^i u + b^i v + c^i = 0$ $(i = 1, 2, 3, 4)$ are given; and also four arbitrary lines, of which no three are concurrent, with the equations $\alpha^i x + \beta^i y + \gamma^i = 0$ $(i = 1, 2, 3, 4)$, we can let these points and lines correspond to each other in a reciprocal transformation by setting

$$\gamma^i(a^i a_1 + b^i a_2 + c^i a_3) - \alpha^i(a^i c_1 + b^i c_2 + c^i c_3) = 0,$$

$$i = 1, 2, 3, 4,$$

$$\gamma^i(a^i b_1 + b^i b_2 + c^i b_3) - \beta^i(a^i c_1 + b^i c_2 + c^i c_3) = 0.$$

These are eight equations with the eight unknown ratios $\frac{a_1}{c_3}, \frac{a_2}{c_3}, \frac{a_3}{c_3}, \frac{b_1}{c_3}, \frac{b_2}{c_3}, \frac{b_3}{c_3}, \frac{c_1}{c_3}, \frac{c_2}{c_3}$, from which the latter may be found definitely. Hence the theorem: *A quadrilateral and a quadrangle always determine a reciprocal transformation in which they correspond to each other.* The reciprocal transformation is the most general dualistic transformation and includes polar reciprocity as a special case. This is easily recognized by comparing formulas (2) of § 37 and (1) of this section.

2. We shall now determine those lines of the coincident planes (u, v) and (x, y) which coincide with their corresponding points. A line with the coordinates u and v passes through the point with the coordinates x and y if

$$ux + vy + 1 = 0.$$

Hence the points (x, y) whose corresponding lines (u, v) according to (1) pass through them satisfy the condition

$$\frac{a_1x+b_1y+c_1}{a_3x+b_3y+c_3}x+\frac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}y+1=0,$$

or

$$(5) \qquad a_1x^2+(b_1+a_2)xy+b_2y^2+(c_1+a_3)x+(c_2+b_3)y+c_3=0,$$

which represents a real or imaginary conic C. Conversely, for the lines (u, v) whose corresponding points (x, y) lie on them, we have the condition

$$(6) \quad (b_2c_3-b_3c_2)u^2+(b_3c_1-b_1c_3+a_3c_2-a_2c_3)uv+(a_1c_3-a_3c_1)v^2+ \\ (b_1c_3-b_2c_1+a_2b_3-a_3b_2)u+(a_2c_1-a_1c_2+a_3b_1-a_1b_3)v+ \\ (a_1b_2-a_2b_1)=0,$$

which represents a conic of the second class Γ. The conics C and Γ are generally different, as may be seen by applying the results of § 36, 4, to equations (5) and (6).

To every point of C correspond the two tangents from it to Γ; conversely, to every tangent of Γ correspond its two points of intersection with C. If C and Γ have a point P in common, then to P on C correspond two coincident tangents to Γ at P, so that their corresponding points also coincide at P. This is only possible when C and P are tangent at P. From this it follows that *the two conics C and Γ are doubly tangent*, and as there is no distinction analytically, we may say that in case of no real intersections the conics C and Γ have two imaginary tangencies.

3. From (3) it is seen that to a pencil

$$ax+by+c+\lambda(a'x+b'y+c')=0$$

corresponds a range; and the vertex of the pencil corresponds to the line of the range. The converse (apply (4)) is also true. Let now U and S be the points of tangency of C and Γ, and u and s the tangents at U and S, and T their point of intersection, and consider the planes of (u, v) and (x, y) as made up of lines and points and points and lines respectively; i.e., to a couple (u, v),

$(au+bv+1=0)$ in one plane corresponds dualistically a couple (x, y), $(\alpha x+\beta y+1=0)$ in the other plane, and conversely.

No matter whether we consider T as belonging to one or the other plane, SU is the corresponding line in both cases. In the reciprocal transformation T and SU are therefore in the relation of involution. For both conics SU is the polar of T.

The question is whether it is possible to find a reciprocal transformation for which the involutoric property is true in general. For this purpose consider a point

$$au+bv+c=0$$

in the plane (u, v) and the same point $\left(\frac{a}{c}\right), \left(\frac{b}{c}\right)$ in the plane (x, y).

To the point $\left(\frac{a}{c}, \frac{b}{c}\right)$ in the xy-plane corresponds the line with the coordinates

$$(7) \qquad u=\frac{aa_1+bb_1+cc_1}{aa_3+bb_3+cc_3}, \quad v=\frac{aa_2+bb_2+cc_2}{aa_3+bb_3+cc_3} \text{ in the } uv\text{-plane.}$$

To the point $(au+bv+c=0)$ in the uv-plane corresponds the line

$$(aa_1+ba_2+ca_3)x+(ab_1+bb_2+cb_3)y+(ac_1+bc_2+cc_3)=0$$

in the xy-plane. Its line-coordinates are

$$(8) \qquad u'=\frac{aa_1+ba_2+ca_3}{ac_1+bc_2+cc_3}, \quad v'=\frac{ab_1+bb_2+cb_3}{ac_1+bc_2+cc_3}.$$

For an involutoric relation the two lines (7) and (8) must be identical. This will be the case when $b_1=a_2$, $c_1=a_3$, $c_2=b_3$; i.e., if the transformation (1) has the form

$$(9) \qquad u=\frac{a_1x+b_1y+c_1}{c_1x+c_2y+c_3}, \quad v=\frac{b_1x+b_2y+c_2}{c_1x+c_2y+c_3}.$$

According to (2), § 37, these are the formulas for a transforma-

tion by reciprocal polars. To prove this directly the equation (5) of the conic C now becomes

$$(10) \qquad a_1x^2+2b_1xy+b_2y^2+2c_1x+2c_2y+c_3=0.$$

To a point $ux_1+vy_1+1=0$ now corresponds the line (according to (4))

$$(11) \quad (a_1x_1+b_1y_1+c_1)x+(b_1x_1+b_2y_1+c_2)y+(c_1x_1+c_2y_1+c_3)=0.$$

This, however, is the polar of the point (x_1, y_1) with respect to (10).

An involutoric recipcocal transformation is therefore a transformation by reciprocal polars.

In this case the conics C and Γ coincide.

4. The line-coordinates given in (7) and (8) are also identical if corresponding numerators and denominators are proportional. Designating the proportionality factor by λ, these conditions assume the form

$$(12) \qquad \begin{cases} a_1(1-\lambda)a+(b_1-\lambda a_2)b+(c_1-\lambda a_3)c=0, \\ (a_2-\lambda b_1)a+b_2(1-\lambda)b+(c_2-\lambda b_3)c=0, \\ (a_3-\lambda c_1)a+(b_3-\lambda c_2)b+c_3(1-\lambda)c=0; \end{cases}$$

but consistency of these equations requires the vanishing of their determinant, or

$$(13) \qquad \begin{vmatrix} a_1(1-\lambda) & b_1-\lambda a_2 & c_1-\lambda a_3 \\ a_2-\lambda b_1 & b_2(1-\lambda) & c_2-\lambda b_3 \\ a_3-\lambda c_1 & b_3-\lambda c_2 & c_3(1-\lambda) \end{vmatrix}=0.$$

This is the case, first when $b_1=a_2$, $c_1=a_3$, $c_2=b_3$, and $\lambda=1$, as discussed under 3; secondly, by expanding the determinant according to ascending powers of λ and solving the cubic in λ. Thus three values for λ are obtained which make the determinant vanish. One of these values is always real, so that there is at least one real line which with its corresponding point forms an involutoric couple.

To push the investigation of involutoric reciprocity one step further we may put the condition for the equality of u and u', and v and v', in (7) and (8) in the form

$$(14)\quad (a_1x+a_2y+a_3)(a_3x+b_3y+c_3)-(a_1x+b_1y+c_1)(c_1x+c_2y+c_3)=0,$$

$$(15)\quad (b_1x+b_2y+b_3)(a_3x+b_3y+c_3)-(a_2x+b_2y+c_2)(c_1x+c_2y+c_3)=0,$$

where $x=\frac{a}{c}$, $y=\frac{b}{c}$ are the coordinates of the original point.

There are generally four solutions of (x, y) which satisfy (14) and (15) simultaneously. Of these, one is the point of intersection of the lines $a_3x+b_3y+c_3=0$ and $c_1x+c_2y+c_3=0$, which, however, is to be excluded. In fact, according as this point is considered as belonging to one or the other plane, (u, v) or (u', v'), the lines through the origin with the slopes $\frac{u}{v}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, $\frac{u'}{v'}=\frac{a_1x+a_2y+a_3}{b_1x+b_2y+b_3}$ correspond to it. Hence, as we have found before, there are in general only three involutoric pairs in a reciprocal transformation; they are determined by the three remaining points of intersection of (14) and (15) and form a triangle UST, according to 3, in which TU and TS correspond to the points U and S, and T to the line US, involutorically. Hence in a reciprocal transformation there is generally only one involutoric pair (T, US) which is not coincident.

Suppose that this be true for a second pair of this kind, then (14) and (15) would have a fifth common solution which is only possible when the two are identical. Hence the theorem:

If a reciprocal transformation contains two non-coincident involutoric pairs, then all its pairs are involutoric; the transformation is a so-called polar reciprocity.

By two non-coincident involutoric pairs the polar reciprocity is fully determined.

To prove this last theorem equations (9) and the equation for $\frac{u}{v}$ obtained from them may be written in the form

$$\frac{a_1}{c_3}x+\frac{b_1}{c_3}y+\frac{c_1}{c_3}(1-ux)-\frac{c_2}{c_3}uy-u=0,$$

$$\frac{b_1}{c_3}x+\frac{b_2}{c_3}y-\frac{c_1}{c_3}vx+\frac{c_2}{c_3}(1-vy)-v=0,$$

$$\frac{a_1}{c_3}vx+\frac{b_1}{c_3}(vy-ux)-\frac{b_2}{c_3}uy+\frac{c_1}{c_3}v-\frac{c_2}{c_3}u=0.$$

Giving (x, y) two arbitrary values and (u, v) correspondingly two arbitrary values, six equations with the only five unknown quantities $\frac{a_1}{c_3}, \frac{b_1}{c_3}, \frac{b_2}{c_3}, \frac{c_1}{c_3}, \frac{c_2}{c_3}$, are obtained. Designating the two pairs by (x_1, y_1), (x_2, y_2) and (u_1, v_1), (u_2, v_2), the determinant of the six equations becomes

$$\begin{array}{r|cccccc|}
v_1 & x_1 & y_1 & (1-u_1x_1) & 0 & -u_1y_1 & -u_1 \\
-v_2 & x_2 & y_2 & (1-u_2x_2) & 0 & -u_2y_2 & -u_2 \\
-u_1 & 0 & x_1 & -v_1x_1 & y_1 & (1-v_1y_1) & -v_1 \\
u_2 & 0 & x_2 & -v_2x_2 & y_2 & (1-v_2y_2) & -v_2 \\
+1 & v_1x_1 & (v_1y_1-u_1x_1) & v_1 & -u_1y_1 & -u_1 & 0 \\
-1 & v_2x_2 & (v_2y_2-u_2x_2) & v_2 & -u_2y_2 & -u_2 & 0
\end{array}=0$$

In fact multiplying the six rows successively by v_1, $-v_2$, $-u_1$, u_2, $+1$, -1, as indicated, after this multiplication, the sum of the first four rows is equal to the sum of the last two, which shows that any of the six equations may be expressed in terms of the five remaining ones. The above five quantities are therefore uniquely determined, which proves the theorem.

5. In a polar reciprocity, or simply in a polar system, two pairs A, a and P, p determine at once a third. Indeed the line c joining A and P is the polar of the point of intersection C of a and p. The pole of AC is the intersection of a and c, say B. Thus, starting with two pairs, we have constructed a triangle ABC, whose vertices are the poles of its opposite sides. Such a triangle is called a *self-polar triangle* (§ 14). Clearly in every polar system there are an infinite number of self-polar triangles; but by such a triangle a polar system is not completely determined.

To do this, another pair, like P, p, must be added to the given triangle.

6. Without following the subject of polar systems further we remark that the great geometer von Staudt has made it the backbone of his geometry of position. In this connection conics appear as special properties of polar systems and no distinction, or separate treatment of real and imaginary elements, is necessary.

In view of the various methods applied in this work, we have found it advisable to be satisfied with the foregoing short account.

It would be very valuable if some geometer could show how, with polarity as a base, projective geometry might be made as simple and as accessible to the applications as the traditional methods.

§ 39. Theorems of Pascal and Brianchon.[1]

1. Assume six points A, B, C, D, E, F in any order on a conic and consider any two of them, say A and C, as vertices of pencils of rays in the conic, Fig. 49. Then

$$(A \cdot BCDEF) = C \cdot BCDEF).$$

Cutting these pencils by the lines ED and EF respectively, two projective point-ranges,

$$(B_1C_1D_1E_1F_1) = (B_2C_2D_2E_2F_2),$$

are obtained, and as E_1 is identical with E_2 it follows (§ 9) that the two ranges are perspective. Hence B_1B_2, C_1C_2, D_1D_2, E_1E_2, F_1F_2 are concurrent at a point B_3. This will be true no matter how the six points may be distributed over the conic, provided the foregoing order of the points is followed. The lines followed in the order $ABCDEF$ form now a closed hexagon,

[1] PASCAL (1623–1662) discovered his theorem when sixteen years of age and called it Hexagramma Mysticum. It appeared first in Pascal's "Conic Sections," which was published in 1640.

BRIANCHON (1785–1864) published his theorem in 1806 in the Journal de l'Ecole Polytechnique, Vol. XIII.

and we may call opposite sides of this hexagon lines which are separated by two adjacent sides of the hexagon, which is all in analogy with the regular hexagon. The pairs of opposite sides intersect at B_1, B_2, B_3, three collinear points. As the six points

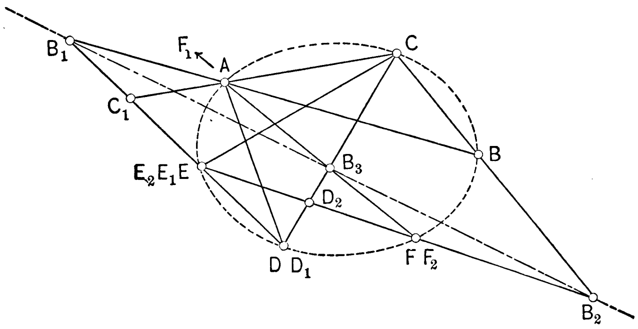


FIG. 49.

were arbitrarily selected, this is generally true, hence PASCAL'S THEOREM:

In any hexagon which is inscribed in a conic, the three pairs of opposite sides intersect in three collinear points.

We shall call such a line of collinearity a *Pascal line.*

By reciprocation, Fig. 50 (§ 37), we obtain immediately in its generality BRIANCHON'S THEOREM:

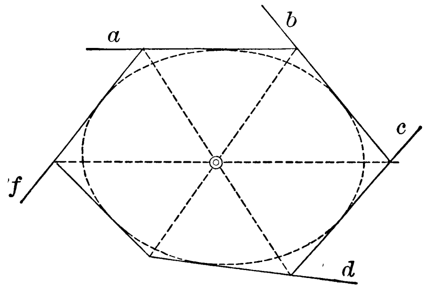


FIG. 50.

In any hexagon which is circumscribed about a conic, the three principal diagonals are concurrent.

We shall call such a point of concurrence a *Brianchon point.*

2. Salmon, in his treatise on Conic Sections, 1848, gave a remarkably simple proof for Pascal's theorem, based upon the abbreviated designation of straight lines in analytic geometry. Let $A=0$, $B=0$, $C=0$, $D=0$, $E=0$, $F=0$ be the equations of the sides of any hexagon inscribed to a conic, and $G=0$ the equation of the straight line joining the vertices $(A=0,\ F=0)$ and $(C=0,\ D=0)$. Then

$$A\cdot C-\lambda B\cdot G=0, \quad F\cdot D-\mu E\cdot G=0$$

are two forms in which the equation of the given conic may be written. From these two forms we get

$$A\cdot C-F\cdot D\equiv G(\lambda B-\mu E).$$

Now the points $(A=0,\ D=0)$ and $(C=0,\ F=0)$ are not situated on the line $G=0$, consequently they must lie on the line $\lambda B-\mu E=0$. In other words, the points $(A=0,\ D=0)$, $(C=0,\ F=0)$, $(B=0,\ E=0)$ are collinear, and as they are the points of intersection of pairs of opposite sides in the hexagon, Pascal's theorem is proved.

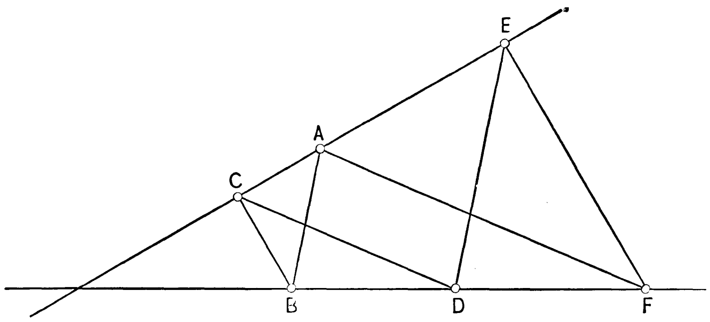


FIG. 51.

By considering $A=0$, etc., as the line-equations of the six vertices of a hexagon circumscribed to a conic, Brianchon's theorem may be deduced in a similar manner.

3. Assuming as a conic a degenerate hyperbola, consisting of two intersecting lines and on each three points, say A, E, C and D, B, F, Pascal's theorem still holds; i.e., the points B_1, B_2, B_3 are collinear.

If $AB \parallel DE$ and $EF \parallel CB$, then B_1 and B_2 and consequently also B_3 are infinitely distant; i.e., also $CD \parallel AF$, Fig. 51. Hence the special theorem:

If on each of two intersecting lines three points A, C, E and B, D, F are chosen, so that AB is parallel to DE and EF parallel to BC, then CD is also parallel to AF.

In this special form Hilbert in his Foundations of Geometry,[1] p. 28, uses Pascal's theorem to establish a non-Archimedean geometry.

Ex. 1. Prove Pascal's special theorem directly.

Ex. 2. Establish the dualistic of Ex. 1.

Ex. 3. If $A=0$, $\gamma A+bB=0$, $\gamma B'+aA'=0$, $A'=0$, $aA'+\gamma' B=0$, $\gamma' A+bB'=0$ (where a, b, γ, γ' are numerical factors and A, B, A', B' linear expressions in x and y) are the sides of a hexagon, prove that this hexagon is inscribed to the conic

$$aAA'-bBB'=0,$$

and that

$$\gamma\gamma' A-abA'=0$$

is the Pascal line of the hexagon. (Bobillier, 1828.)

Ex. 4. If six points on a conic are given, it is possible to pass in five different ways from any point to the others. From each of these four different paths, not chosen before, may be taken to join the remaining points; from each of these three different paths may be selected; and so forth. Finally the original point is reached in $5\cdot4\cdot3\cdot2\cdot1=120$ ways; but as each closing side is contained in one of the original paths, it is evident that only $120:2=60$ different closed hexagons can be formed. *Hence with six points on a conic may be formed sixty different hexagons and consequently sixty different Pascal lines.*

---

[1] *Grundlagen der Geometrie*, Teubner, Leipzig, 1899.

Between these lines exist a number of interesting relations.[1]

Verify the following propositions in a regular hexagon:

The sixty Pascal lines intersect each other three by three in twenty points G (Steinerian points). (Steiner's theorem.)

Besides these points G, the sixty Pascal lines have, three by three, sixty other points H in common. (Kirkmann's theorem.)

There are twenty lines g each of which contains a point G and two points H. Four by four of these lines pass through fifteen points J. (Cayley's theorem.)

The points G lie four by four in fifteen straight lines J. (Steiner's theorem.)

Designating the original six points by 123456, then a Steinerian point is given by the intersection of the Pascal lines of the three hexagons 123456, 143652, 163254.

Ex. 5. State the dualistic of the foregoing theorems.

§ 40. Applications of Pascal's and Brianchon's Theorems.

1. *Construction of a conic when five of its points are given.*

The practical importance of Pascal's and Brianchon's theorem lies in the possibility of constructing an unlimited number of points and tangents of a conic, when five of its determining elements are given.

Let $ABCDE$ be five points of a conic and AB, BC, CD, DE four consecutive sides of an inscribed hexagon. In Fig. 52, it is clear that the Pascal line p passes through B_1, the point of intersection of AB and DE. Now there are an infinite number of points F possible on the conic and consequently an infinite number of Pascal lines through B_1. Thus to every point F on the conic corresponds one Pascal line through B_1. Hence, assuming any line p through B_1, the line EF passes through the intersection B_3 of BC and p. In a similar manner the line FA is obtained by joining A with the point of intersection B_2 of CD with p.

[1] See SALMON-FIEDLER, *Analytische Geometrie der Kegelschnitte*, Vol. II, pp. 459–466, 5th edition.

The point where the produced lines of EB_3 and AB_2 meet is evidently the required point F on the conic, corresponding to the

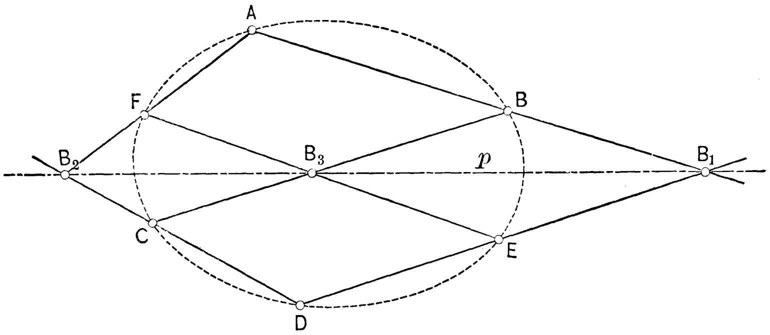


FIG. 52.

chosen Pascal line p. Repeating the same construction for every line p through B_1, all points of the conic are obtained.

To construct the tangent at any point of the conic, say A, consider F infinitely close to A. Apply the general construction of p for $ABCDEF$, then the line joining B_2 with A is the tangent at this point.

2. *Construction of a conic when five of its tangents are given.*

Let a, b, c, d, e be the given tangents, Fig. 53, forming five consecutive sides of a circumscribed hexagon. The line b_1 joining the points of intersection of a and b, and d and e, passes through the Brianchon point P. Now, every sixth tangent determines another point P on b_1. Conversely, every point P on b_1 determines a sixth tangent of the conic. Thus, to find a sixth tangent f, assume any point on b_1 as the Brianchon point P. Then the line through bc and P will be the line b_2 cutting e in the point where also f cuts. In a similar manner, the line joining the point of intersection of c and d with P is b_3, which, when produced, cuts a in the same point as f. Hence the line joining the points of intersection of e and b_2, and b_3 and a, is the sixth tangent corresponding to the chosen P. Repeating this construction for all points of b_1, all tangents of the conic are obtained.

To construct the point of tangency of any tangent, we may consider this one as two coincident tangents (consecutive), say a and f, and these with the remaining four, when subject to the general construction of the Brianchon point, lead to the required point of tangency.

3. By the same methods conics may also be constructed when they are determined by mixed elements; i.e., points and tangents, always five in number. In these problems a tangent appears as

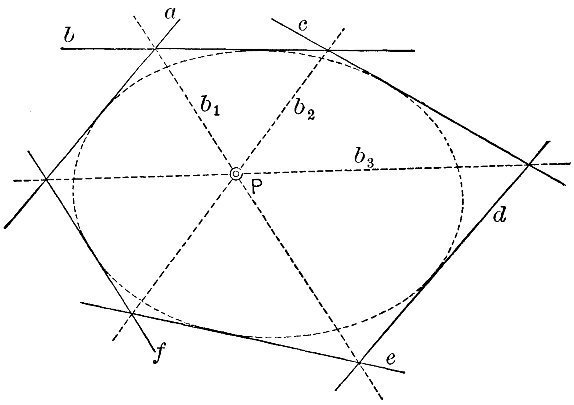


FIG. 53.

a line joining two consecutive points, and a point as the point of intersection of two consecutive tangents. The same constructions may also be extended to cases where one point, two points, or one tangent is infinitely distant.

Ex. 1. Given five points of a conic; to construct the tangents at these points.

Ex. 2. The dualistic of Ex. 1.

Ex. 3. Given three points and the directions of the asymptotes of an hyperbola; to construct any number of points of the hyperbola.

Ex. 4. Given four tangents of a parabola (one tangent is infinitely distant). To construct any number of its points.

Ex. 5. Given four points and a tangent of a conic; construct other points of the conic.

Ex. 6. Dualistic of Ex. 5.

Ex. 7. Given three points and two tangents of a conic. To construct it. Also make the dualistic construction.

Ex. 8. Given three points, a tangent, and its point of tangency; construct the conic.

Ex. 9. Given the two tangents at the given vertices of an ellipse or hyperbola and a third tangent; to construct any number of tangents.

Ex. 10. Given the two asymptotes and a tangent of an hyperbola; to construct it.

Ex. 11. Given the axis, vertex, and two other points of a parabola; construct it.

Ex. 12. Given three points and an asymptote of an hyperbola; to construct it.

§ 41. Conics in Mechanical Drawing and Perspective.

1. *To inscribe an ellipse in a parallelogram.*

The middle points of the sides shall be the points of tangency of the ellipse. Two points of tangency may be designated by AB and CD, and the third by E, Fig. 54. The explanation of the construction of points of the ellipse by Pascal's theorem is identical with that of Fig. 52, § 40, and is apparent from Fig. 54. By assuming a second Pascal line through L with points H and J corresponding to M and N on the first Pascal line, a second point G is obtained. The same construction repeated for other Pascal lines through L gives further points of the ellipse, so that the ellipse through these points may be sketched free-hand or by mans of a curved ruler. In this figure the ellipse appears manifestly as the product of two projective pencils with A and E as vertices. In fact,

$$(AMCH) = (KNCJ),$$

since these points are projected by one and the same pencil through L. Taking a Pascal line parallel to KC and desig-

nating its point of intersection with AC produced by I, then the point corresponding to J has moved to infinity on KC, and

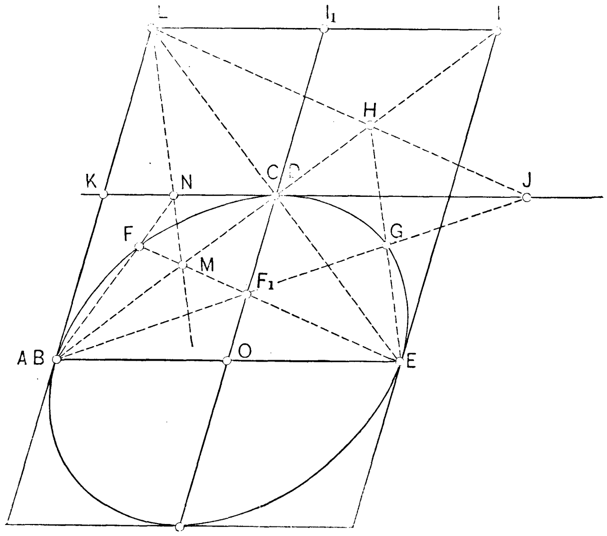


FIG. 54.

to the Pascal line LI corresponds the point E on the ellipse. Now

$$(AMCI) = (KNC\infty),$$

and as $AC = CI$, these ratios become

$$\frac{1}{2} \cdot \frac{AM}{CM} = \frac{KN}{CN}.$$

But there is also

$$(AMCI) = (OF_1C\infty);$$

hence

$$\frac{KN}{CN} = \frac{OF_1}{CF_1}.$$

From this it follows that the rays AN and EM of the pencils through A and E trace on KC and OC similar point-ranges. If, therefore, KC and OC are divided into any number of equal parts and the division-points are numbered from K to C and from O to C, then the rays joining E and A with equal numbers on KC and OC intersect each other in points of the ellipse. In a similar way this construction may be extended to the remaining three quarters of the ellipse. The same method may obviously be applied to rectangles and squares. See Figs. 37, 38; § 27.

2. *To inscribe an ellipse to any quadrilateral.*

A quadrilateral may be considered as the perspective of a square, and it must therefore be possible to apply the previous construction to any quadrilateral. The distances KC and OC, Fig. 55, must now be divided perspectively into a number of equal parts. The fundamental principle of perspective division is the following:

If KC as a side of a rectangle $AKCO$, in perspective, shall be divided into two equal parts, draw the diagonals AC and KO and join their point of intersection W to the point X, where AK and OC produced meet. WX cuts KC at its middle point, M.

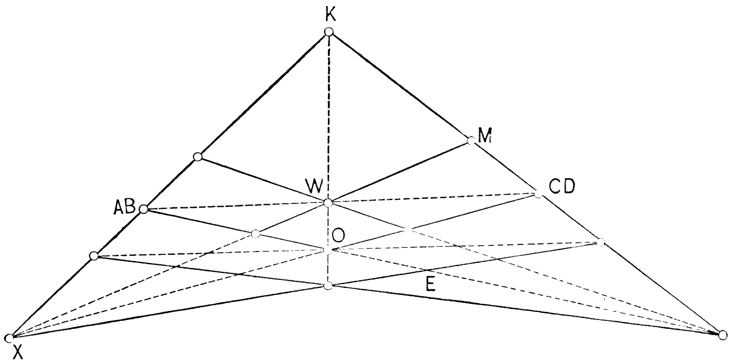


Fig. 55.

In the first place, the points AB, C, E, etc., were obtained by the application of this principle to the given quadrilateral.

By the same principle KM and CM may be again bisected.

OC may be subdivided in the same manner. Fig. 38, § 27, illustrates the construction of an ellipse inscribed in a quadrilateral by this principle.

The problem to inscribe an ellipse into a quadrilateral appears in a great number of special forms in perspective. For example, a trapezoid may be considered as the perspective of a square having two opposite sides parallel to the picture-plane, as in the case of window-frames and doors.

3. *To construct a parabola having the vertex, the major axis, and a point given.*

Let, in Fig. 56, the vertex be designated by AB, the infinitely distant point of the axis by DE, and the third point by C. Evidently any line p parallel to the tangent at the vertex may be considered as a Pascal line. The construction

$$\left.\begin{matrix} AB \\ DE \end{matrix}\right\} L, \quad \left.\begin{matrix} BC \\ EF \end{matrix}\right\} M, \quad \left.\begin{matrix} CD \\ FA \end{matrix}\right\} N$$

for the assumed Pascal line p gives us a point F of the parabola. If p varies, the point-ranges traced by M, N, F_1 on AC, KC, AK are all similar. Hence, dividing KC and AK in any number of equal parts, numbering the division-points from K to C and from A to K, a line joining A to any number on KC and a line through the equal number on AK, parallel to the axis, cut each other in a point of the parabola.

4. *Construction of a parabola which is the funicular polygon of a uniformly distributed load on a horizontal beam.*

If a load is uniformly distributed on a horizontal beam, then the funicular polygon is a parabola limited by points in perpendiculars through the extremities of the beam. The tangents at the extremities of the parabola are known; they are parallel to the extreme lines of the force polygon. Designate in Fig. 57 the tangents at the extremities by ab and de, and the infinite tangent by c. Then the Brianchon points P are situated on ad, and lines through P parallel to ab and de (the tangents at the extremities) cut these in two points x and y through which the

sixth tangent f passes. If P moves on ad, then the points x and y trace on $ab \cdot Q$ and $de \cdot Q$ two equal point-ranges. Hence, dividing

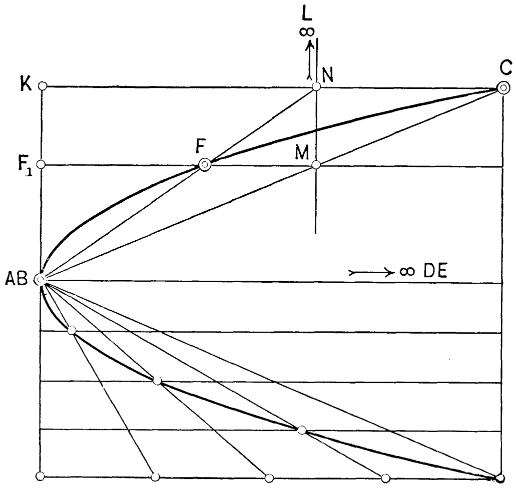


FIG. 56.

these distances in a number of equal parts and numbering them from ab and Q, the lines joining equal numbers are tangents of the required parabola. To find the point of tangency of f, replace ab by c, de by f, f by ab, and the infinite tangent by de,

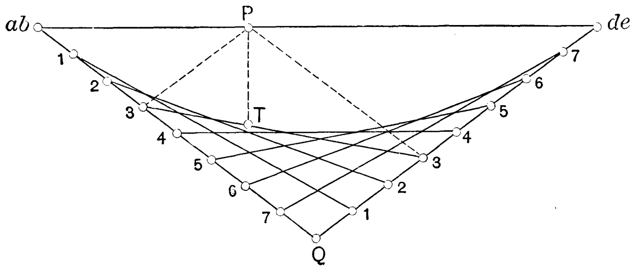


FIG. 57.

de being the infinite point of the axis of the parabola. For this arrangement P is also the Brianchon point, and the construction shows at once that the required point of tangency T is cut out by a line through P parallel to the axis of the parabola.

To sum up, we have the following construction for a parabola touching the sides of an isosceles triangle ABC, $AC=BC$, at A and B: *Divide AB, BC, and CA into the same number of equal parts and number the division-points from A to B, from A to C, and from C to B. The lines joining equal numbers on AC and CB are tangents of the required parabola, and the perpendiculars from corresponding equal division-points on AB cut these tangents in their points of tangency.*

5. *Construction of an equilateral hyperbola when its asymptotes and the tangent at a vertex are given.*

In Fig. 58 designate the asymptotes by ab and de, and the tangent at the vertex by c. Let c cut the asymptotes at A and B.

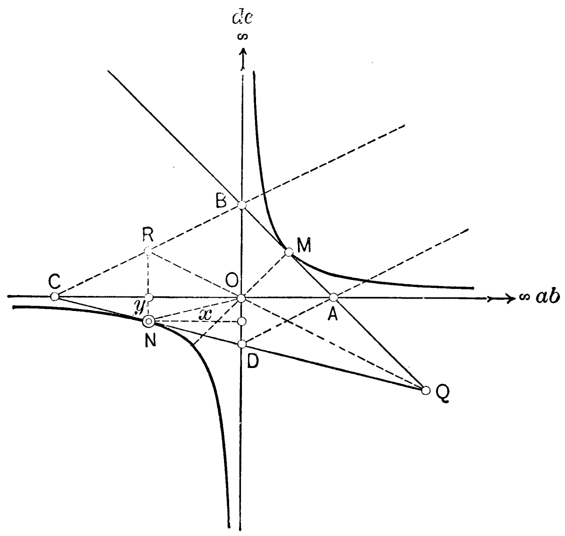


FIG. 58.

In this case the Brianchon points are infinitely distant. Hence, drawing through A and B two parallel lines in any direction, cutting the asymptotes at C and D, the line joining C with D will be a required tangent of the hyperbola. To study the metrical relations of this hyperbola, we have

$$\triangle AOD \backsim \triangle COB,$$

hence $$DO:AO=BO:CO,$$

or $$CO\cdot DO=AO\cdot BO=\text{constant}.$$

Designating the distance of the vertex M from the asymptotes by k, there evidently is

$$CO\cdot DO=4k^2,$$

a relation which holds for any tangent. *Hence the triangle between the asymptotes and any tangent has a constant area.* (This is true for any hyperbola, as might be proved in a similar manner.)

To find the point of tangency of CD, replace the designation f by ab, ab by de, de by c, and AB by f. (The student should make a new figure.) Join the point of intersection Q of AB and CD to O and produce to the point of intersection R with BC; then R is the new Brianchon point and the line through R parallel to OD cuts CD in the required point of tangency N. As $ABCD$ is a quadrilateral in which $AD \parallel BC$ and O and Q are diagonal points, $BR=RC$, hence also $DN=CN$. *The point of tangency bisects, therefore, the tangent between the asymptotes (general proposition).*

Designating the coordinates of N by x and y, we have $x=\frac{1}{2}CO$, $y=\frac{1}{2}DO$; hence $xy=\frac{1}{4}CO\cdot DO$, and as $CO\cdot DO=4k^2$,

$$xy=k^2.$$

This is the equation of the hyperbola referred to its asymptotes.

A full treatment of this case was given in view of its importance in the graphical representation of Boyle's law expressing the relation of the volume x and the pressure y of a gas.

§ 42. Special Constructions of Conics by Central Projection and Parallel Projection.[1]

1. *Given five points of a conic, to construct a circle of which the given conic is a perspective.*

[1] For the collection of these problems the author is indebted to Dr. Karl Doehlemann's Geometrische Transformationen, I. Teil, Göschen, Leipzig, 1902.

In §§ 20 and 27 it has been shown analytically and synthetically that every quadrilateral may be considered as the perspective of a rectangle which is always inscribed to a certain circle. It is therefore possible to construct four points A, B, C, D of a rectangle as the points whose perspectives are four given points A', B', C', D' of the conic. As has been explained in § 27, the two diagonal points M' and N' determine the vanishing-line and are the vanishing-points of the pairs of parallel sides AB, CD and AD, BC. The center of perspective collineation is situated on a circle over $M'N'$ as a diameter, and AB, CD and AD, BC are respectively parallel to SM' and SN', Fig. 59. The axis of col-

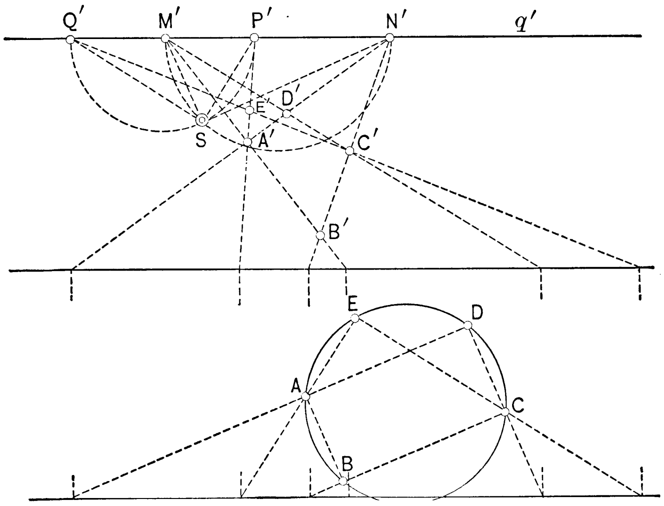


FIG. 59.

lineation s must be chosen parallel to q' or $M'N'$ at any distance from it. From S and s, $ABCD$ is perfectly determined. To determine its position in space the distance-circle with S as a center must be given. There are therefore three elements, S, s, and distance-circle, which determine $ABCD$, of which $A'B'C'D'$ is a perspective, completely. Hence there are ∞^3 rectangles in space of which $A'B'C'D'$ is a perspective. If now E, of which E' is the perspective, shall also be situated on the circle through

$ABCD$, notice that AC is a diameter, hence AEC a right angle. Consequently if we produce $A'E'$ and $C'E'$ to their intersections P' and Q' with q', the center S necessarily also lies on the circle over $P'Q'$ as a diameter.

In the figure the construction of $ABCDE$ has been removed parallel to s in order to make it clearer. In this construction we may dispose arbitrarily of s and of the distance-circle. Hence there are ∞^2 circles in space which may be transformed into a given conic by perspective, under the given conditions. To make this proposition general it must be remembered that the analytic expression for perspective involves three essential parameters. If a translation of the center of perspective is added, two more conditions enter, so that, together with the choice of the distance-circle, six constants perfectly determine a central projection. If, therefore, the general equation

$$ax^2+2bxy+cy^2+2dx+2ey+f=0,$$

by means of this projection, is transformed into

$$Ax^2+2Bxy+Cy^2+2Dx+2Ey+F=0,$$

this equation contains those six constants. If this equation shall represent a circle, the conditions $A=C$, $B=0$ must be satisfied, so that of the six constants only four remain independent.

Hence the theorem:

A conic may be considered as central projection of ∞^4 circles in space.

Ex. 1. Given five tangents of a conic; to construct a circle of which the given conic is a perspective.

Hint: Any four of the given tangents may be transformed into a rhomb circumscribed to the required circle. The diagonals of this rhomb are perpendicular and intersect at the center M of the required circle. Furthermore, the piece of the fifth tangent between two parallel sides of the rhomb appears under a right angle from M.

Ex. 2. *Any two conics in a plane may be considered as the central projection of two circles.* (Monge.)

The two circles are supposed to be in one and the same plane. Now every point of the plane may be taken as the center of projection, so that there are $\infty^3 \cdot \infty^3 = \infty^6$ central projections of a plane. Transforming the equations of the given conics, six parameters are introduced of which we may dispose arbitrarily. In order that the transformed equations represent circles, four conditions must be satisfied, so that there is still an infinite number of possibilities for the problem left.

In case that the given conics have four real points of intersection, imaginary elements are introduced in the solution. The validity of the geometrical problem in this case is maintained by *Poncelet's principle of continuity.*[1]

Ex. 3. Prove that any conic and a straight line in its plane may be projected centrally into a circle and the infinite line of its plane.

2. *Conics as intersections of right cones.*

Let in a plane perpendicular to the paper, Fig. 60, a conic K with the foci F, F_1 and the vertices A, A_1 be given. At one of the foci, say F, construct any sphere, S, tangent to the plane of the conic, and from A and A_1 draw, in the plane of the paper, two tangents to this sphere, intersecting at V. Consider V as the vertex of a cone tangent to S; this cone will be a right cone cutting the plane of K in a certain conic K' with the same vertices A and A_1. Let the cone touch the sphere along the circle whose plane is T and which cuts the plane of K in a line perpendicular to the plane of the paper. This line appears as a point D. Assume any point P' on K' and let $P'V$ cut T in Q; then $P'Q = P'F$ (in space). The true length of $P'F$ is $P'R$, which is parallel to VA. Now, no matter where P' is taken on K', $P'R/P'D = P'F/P'D =$ constant. This constant is also equal $AB/AD = AF/AD$. Hence, as P' is the locus of the points whose distances from a fixed point F and a fixed line (D) have a constant ratio, it must

[1] Stated by Poncelet in the introduction of his *Traité*. It consists in the assumption that if one figure is obtained from another figure by a continuous variation, then projective properties derived from the first figure also hold for the second figure. The principle, however, is rigorous only when proved analytically.

be a conic with the focus F and the directrix (D) and is therefore identical with the original conic K.

In Fig. 60 K has been assumed as an ellipse. For every sphere tangent at F there is consequently a right cone tangent

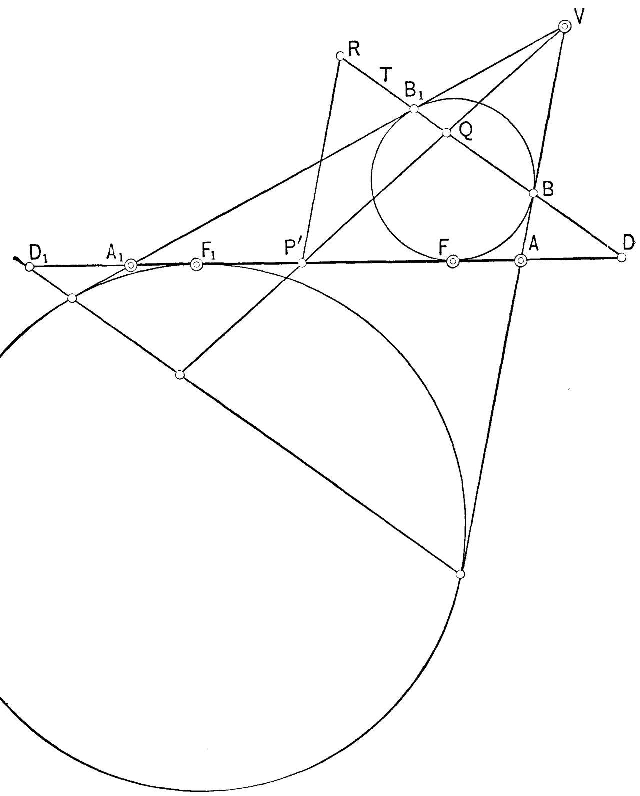


FIG. 60.

to it and of which the given ellipse is a section. Now

$$VA_1 - VA = VB_1 + B_1A_1 - VB - BA = B_1A_1 - BA = A_1F - AF = FF_1;$$

i.e.,
$$VA_1 - VA = FF_1 = \text{constant}.$$

V moves, therefore, on an hyperbola having A, A_1 as foci and F, F_1 as vertices.

If, in place of an ellipse, an hyperbola is chosen for K, V will be on an ellipse having the foci of the hyperbola as vertices and its vertices as foci.

To sum up we have the theorem:

The locus of the vertices of all right cones passing through a given ellipse is an hyperbola having the vertices of the ellipse as foci and its foci as vertices, and whose plane is perpendicular to the plane of the ellipse.

The locus of the vertices of all right cones passing through a given hyperbola is an ellipse whose vertices and foci coincide with the foci and vertices of the ellipse, and whose planes are perpendicular to each other.

Ex. Prove that the locus of the vertices of all right cones passing through a given parabola is a parabola having the vertex of the first as a focus and the focus as a vertex. The planes of the two parabolas are perpendicular.

That there are no other right cones in these problems with the enumerated properties follows from the fact that in every right cone and one of its plane sections there is only one plane of symmetry with respect to the conic section. Conversely, if a conic is given, the vertex of a right cone can only be in this plane of symmetry, the plane passing through the foci and perpendicular to the plane of the conic.

3. *Perspective between any two given conics.*

Let $K=0$ and $K'=0$ be the equations of any two conics in the same plane. Apply a general perspective collineation to K, thus introducing five arbitrary parameters into the transformed equation. In order to make this last equation identical with $K'=0$, corresponding coefficients must be set equal. This gives five equations between the five parameters of the perspective collineation, and as these equations are of the second degree there will be several solutions of the problem. Two conics in a plane may therefore always be considered as perspectives of one another. Without discussing the possibilities of real and imaginary solutions of these equations the case will be considered where K and K' have four real points of intersection 1, 2, 3, 4, Fig. 61.

K and K' have the self-polar triangle XYZ in common. Designate the four common tangents by I, II, III, IV, and consider the points of intersection S of III and IV, and S_1 of I and II. Evidently the centers of perspective must be sought in such

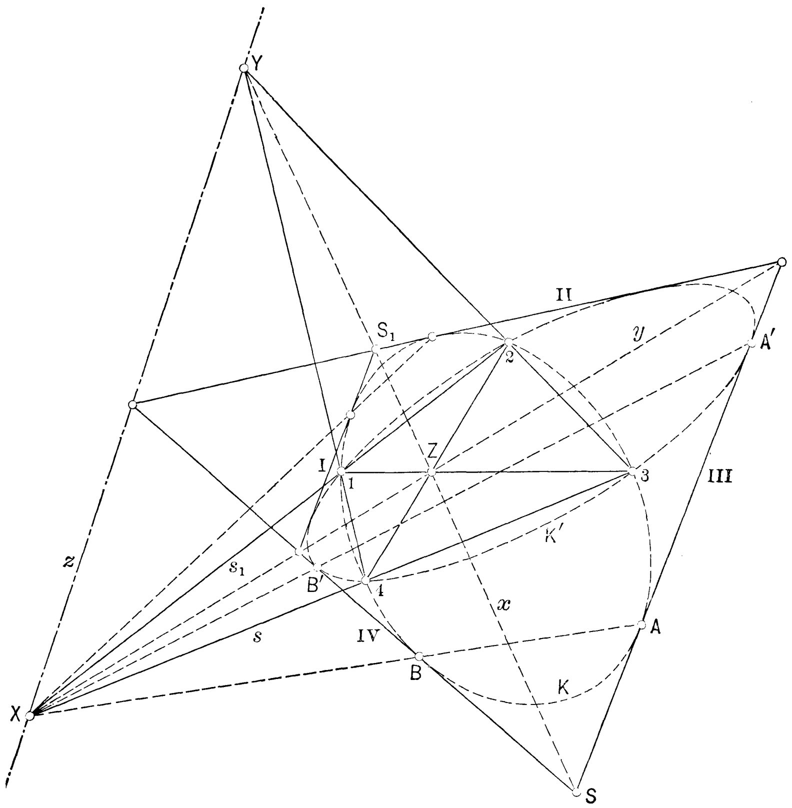


FIG. 61.

points of intersection of common tangents, because a tangent from the center of perspective to one conic is also a tangent to the perspective conic. The common chords 12 and 34 as well as the chords of contact AB and $A'B'$ pass through X when produced. Choosing 34 as the axis of a central collineation, S as

the center, A, A' as corresponding points, then the collineation is perfectly determined, and the conic K is transformed into a conic K'', which, however, is identical with K', since it has the points 3, 4 and the points of tangency A', B'; i.e., six points in common with K'. Instead of s we may also choose the chord s_1 as the axis, and S as the center of a collineation. Hence with S as a center there are two central collineations transforming K into K'. Conversely, every chord, as s, may serve for two collineations with S and S_1 as centers. The same can be analogously proved for every common chord and point of intersection of two common tangents. We have therefore the theorem:

If two conics K and K' have four real points of intersection, then there are 12 central collineations in which K and K' correspond to each other. For every point of intersection of two common tangents there are two chords which may be taken as axes of two of those 12 collineations. Conversely, to every chord belong two points of intersection of common tangents as centers of two such collineations.

These propositions admit of an easy interpretation in space. As every common chord determines two centers of collineation, it follows that *there are two cones through two conics in space with two points in common.*

In § 33 it has been shown that on account of the rectangular polar involution around the center of collineation not being changed, a circle concentric with the center of collineation is transformed into a conic whose focus is in this center. Generally, for the same reason, a conic one of whose foci coincides with the center of collineation is transformed into a conic having the same focus.

But this is also in agreement with the previous result. A focus of a conic may be considered as the point of intersection of two conjugate imaginary tangents from the circular points. Two conics with the same focus have therefore two common imaginary tangents, and their real point of intersection may be assumed as a center of collineation between the two conics.

Ex. 1. Discuss the case and make the construction when

K and K' intersect in two real points and have two parallel tangents.

Ex. 2. Make the construction when K and K' are tangent.

Ex. 3. Discuss the arrangements of K and K' in order to obtain all special cases of perspective collineation.

4. *Given five points of a conic K,—A, B, C, D, E;* through *two of these, say A and B, pass a circle K', and find the center of perspective S for which K and K' are corresponding.*

In Fig. 62 construct the pole X of s or AB, which we assume as the axis of collineation. This is easily done by means of the points of intersection P and Q of CD and DE with s and their polars p and q in the quadrilaterals $ABCD$ and $AEBD$. Construct also the pole X' of s with respect to K', then X and X' are corresponding points in the collineation and the center S

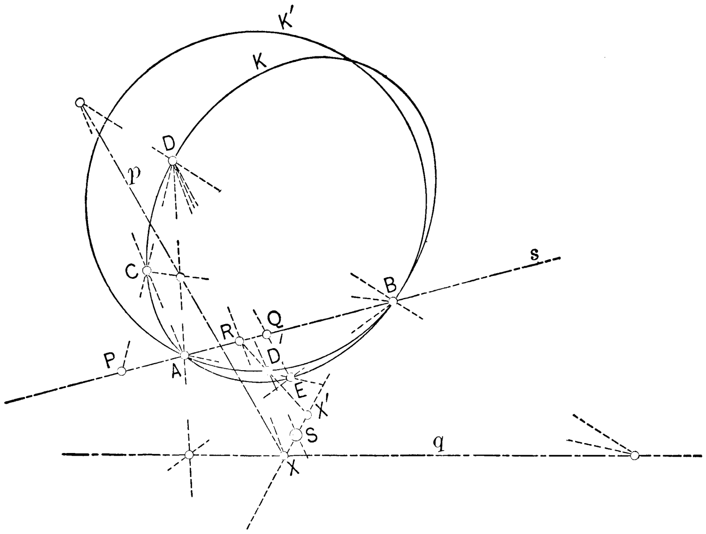


FIG. 62.

must be on the line joining X with X'. DX and $D'X'$ meet on s, in a point R; hence $X'R$ cuts K' in D'. Joining DD' and producing gives on XX' the required center S. Having found S, it is an easy matter to construct further elements of K from the corresponding elements of K'.

A similar problem was solved in the first part of this section.

5. *Given three points A, B, C of a conic K and the tangents at A and B. To find the point of intersection of this conic with a given line g.*

Designate the intersection of the tangents by S, Fig. 63, and draw any circle K' tangent to SA and SB at the points A' and B'. K and K' are now corresponding conics in a collineation with S as a center and A, A'; B, B' as corresponding pairs. SC

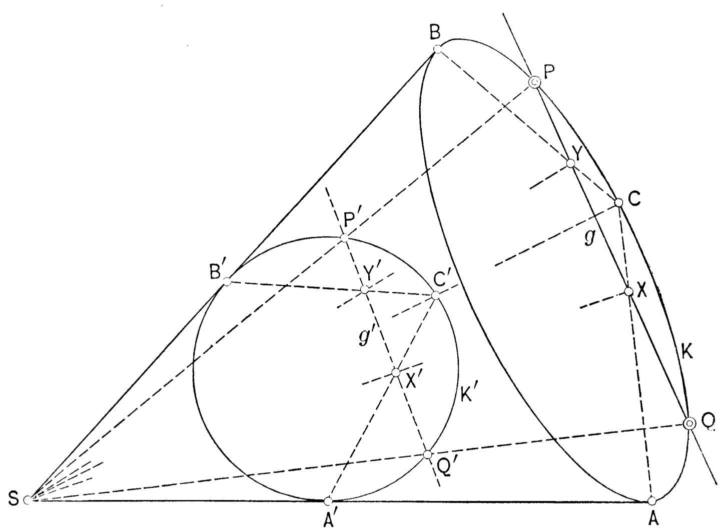


FIG. 63.

cuts K' in C'. Draw CA and CB cutting g in X and Y. Draw SX and SY, which by $C'A'$ and $C'B'$ are cut in X' and Y'. The line joining $X'Y'$ is g'. Let g' cut K' in P' and Q', then SP' and SQ' produced cut g in P and Q, the points of intersection of g with K.

With exactly the same designations we can immediately solve the special case:

Given the asymptotes of an hyperbola and another point. To find the points of intersection of this hyperbola with a given straight line.

Nothing is changed in the previous construction except that A and B are the infinite points on the asymptotes.

6. *To construct a conic K when three points A, B, C and two tangents a and b are given.*

Draw again a circle K' tangent to a and b and assume the intersection S of a and b as the center of collineation, Fig. 64. Join S to A, B, C and designate the points of intersection of SA, SB, SC with K' by A', B', C' and A'', B'', C''. If we let

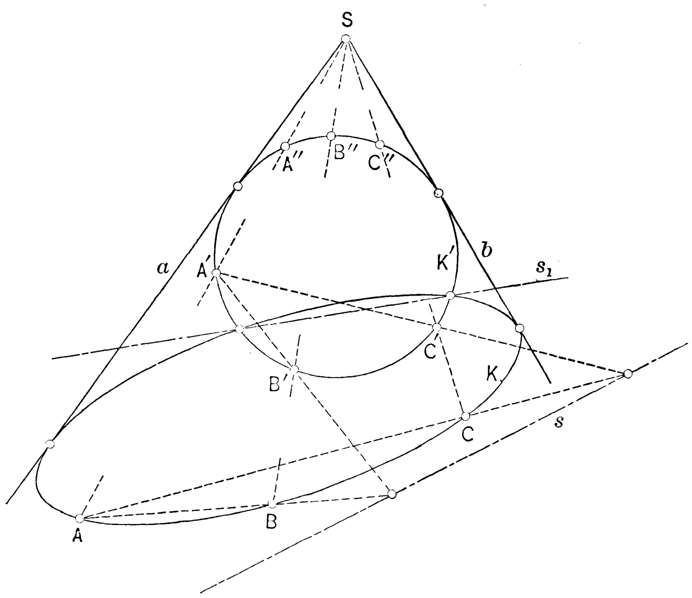


Fig. 64.

A', B', C' correspond to A, B, C, then s is the axis of collineation; if the corresponding points are A'', B'', C'', then s_1 will be the axis of collineation. In both collineations the same conic K corresponds to K'. But we may also let A', B'', C' correspond to A, B, C, which will lead to a different conic K. The arrangement $A''B'C''$, ABC leads to the same conic. There are eight different correspondences possible which in groups of two lead to the same conic. *The problem admits, therefore, of four different solutions.*

7. *Given three points of a conic K,—A, B, C,—and a focus S. To construct the conic.*

Draw in Fig. 65 any circle tangent to the conjugate imaginary tangents from S to K; i.e., draw anv circle K' with S as a center.

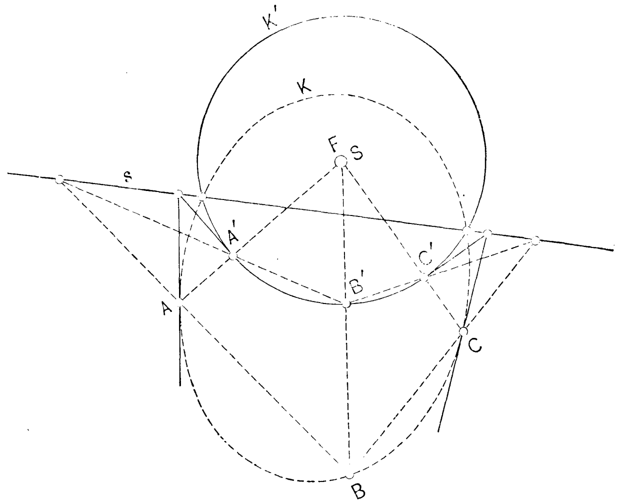


FIG. 65.

The problem and the construction are in this case exactly the same as in problem 6. Here also there are four different solutions.

8. *Given four points and a tangent of a conic, to construct it.*

In Fig. 66 let A, B, C, D be the given points and t the given tangent of the conic K. Consider AB or s as the axis of a collineation, and any circle K' through A and B as the perspective of K. By means of the quadrilateral $ABCD$ construct the polar p of P with respect to K, and also the polar p' of P with respect to K'. p and p' meet in a point of s. From the point where t cuts s draw the tangent t' to K' and let this tangent correspond to t in the collineation. t and t' cut p and p' in two corresponding points X and X', and the center or, if there are several, the centers of possible collineations must be on the line joining X with X'. To determine these centers, join C with X, and the point of intersection of CX with s to X'. Where the last line cuts

K' are two points C' and C_1' which correspond to C. Hence there are two different collineations. Their centers S and S_1 are ob-

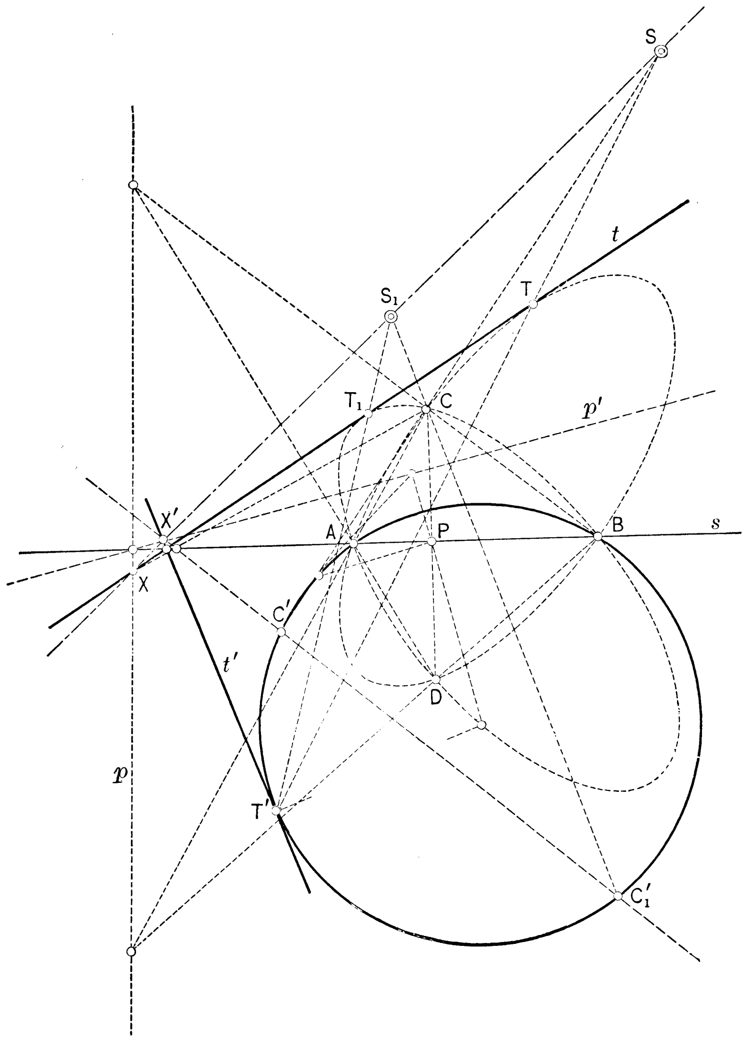


Fig. 66.

tained as intersections of $C'C$ and $C_1'C$ with XX'. To the point of tangency T' of t' with K' correspond in the two collineations T

and T_1 on t, which are the points of tangency of the two conics K passing through $ABCD$ and tangent to t.

That these are the only solutions is not apparent from this construction; it simply shows how conics with the required conditions may be found.

9. *Osculating Circle of a Conic.*

If a conic K passes through the center S of the collineation, then K' will be tangent to K at S. If, furthermore, also the axis s passes through S, one of the remaining two points of intersection of K and K' will coincide with S, and K and K' have at S a contact of the second order. If K is a circle, K will be the osculating circle to K' at S. The remaining fourth point of intersection will be on s. In case that S is on s, the counter-axes of collineation q' and r will be on opposite sides of s. The center M of the circle K is the pole of the infinitely distant line q with respect to K. The corresponding point M' of the collineation is the pole of q' with respect to K'. If now a diameter of K turns about M, the rays joining S with its extremities form a rectangular involution of rays around S which is identical with the involution of rays joining S to the extremities of the chords through M' corresponding to the diameters through M. Hence M' is obtained as the pole of the involution of points on K', which when joined with S give a rectangular involution of rays. q' is the polar of M' with respect to K', and s is a line through S parallel to q'.

It is now possible to solve the problem: *Given five points of a conic K', to construct the osculating circle at any of the given points.*

In Fig. 67 let A, B, C, D, E be the given points, and A the point at which the osculating circle shall be constructed. Join A with B and C; at A erect perpendiculars to AB and AC, and by Pascal's theorem construct the intersections B_1 and C_1 of these perpendiculars with K'. BB_1 and CC_1 cut each other at M'. In the quadrilateral BB_1CC_1 the polar q' of M' is easily found. Through A draw s parallel to q', and find by Pascal's theorem the intersection F of s with K'. K is the circle passing through F and tangent at A to the perpendicular to AM'.

If K is tangent to s at the point S, the osculation will be of the third order; i.e., K and K' will have four points in common at S.

Ex. 1. To construct a conic when the osculating circle at one of its points, A, and two other points, B and C, are given.

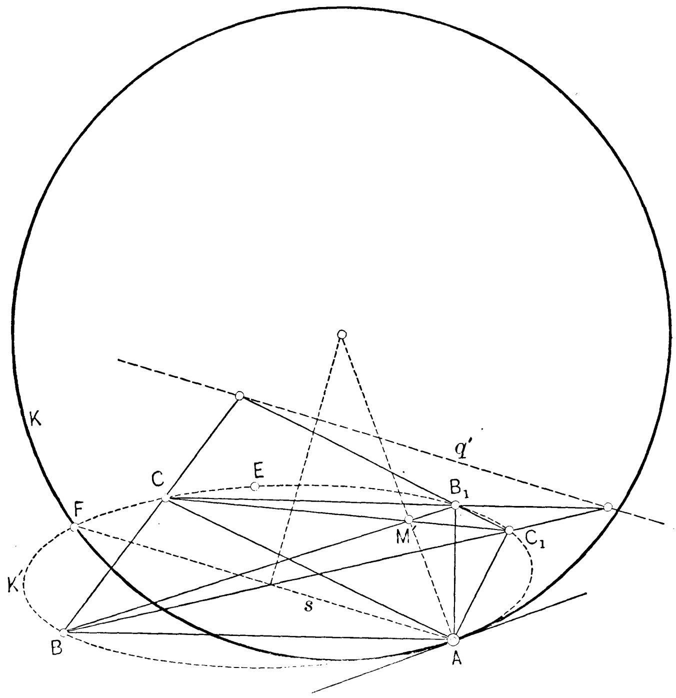


FIG. 67.

Ex. 2. To construct the osculating circle at the vertex of a conic which is determined by major and minor axes.

Ex. 3. To construct a parabola when the osculating circle at its vertex is known.

Ex. 4. Given three tangents and two points of a conic; construct the conic. Dualistic problem of § 42, 6.

Ex. 5. Given four tangents and a point of a conic; construct the conic. Dualistic problem of § 42, 8.

§ 43. Problems of the Second Order.

1. In the previous section and even farther back we have occasionally touched upon problems of the second degree. We shall now pay particular attention to a few geometrical problems which analytically are equivalent with the solution of an equation of the second degree. Most of these problems may be reduced to the problem, *to find the double-points of two coincident projective point-ranges*. This was done analytically in the first chapter. Geometrically we may solve it by the following proposition, which in a little different form appears as Pascal's theorem: Six points, A, B, C, A', B', C', on a conic K determine two projective ranges of points on K, so that for any point P on K we have the projective pencils

$$P(ABC\ldots) = P(A'B'C'\ldots).$$

The pairs of sides AB', $A'B$; BC', $B'C$; CA', $C'A$ intersect in three collinear points, on the Pascal line p. Considering two points, for instance B and B', as carriers of pencils, then

$$(B\cdot A'B'C'\ldots) = (B'\cdot ABC\ldots),$$

and as BB' is a ray common to both pencils, they are perspective and have p as the axis of perspective. Two rays joining B and B' with any point on p cut K in two corresponding points of the projective ranges on K. From this it is clear that *the points of intersection of p with K are the double-points of the projectivity.* These are real, coincident, or imaginary according as p cuts, touches, or does not cut K.

If, instead of six points, six tangents a, b, c, a', b', c' of the conic are given, the lines joining the pairs of intersection ab', $a'b$; bc', $b'c$; ca', $c'a$ all pass through the Brianchon point P. Considering two of the tangents, b and b', and cutting these with the remaining tangents, two perspective ranges

$$(b\cdot a'b'c'\ldots) = (b'\cdot abc\ldots)$$

with the common point bb' (self-corresponding) are obtained. Taking any ray through P and cutting it by b and b', then the tangents to K from these two points are two corresponding tangents. *The tangents from P to K are the double tangents of the projectivity.* We postulate now that a problem is geometrically solvable if it can be solved by compass and ruler. Hence, replacing K by a circle, we can now solve the following problems:

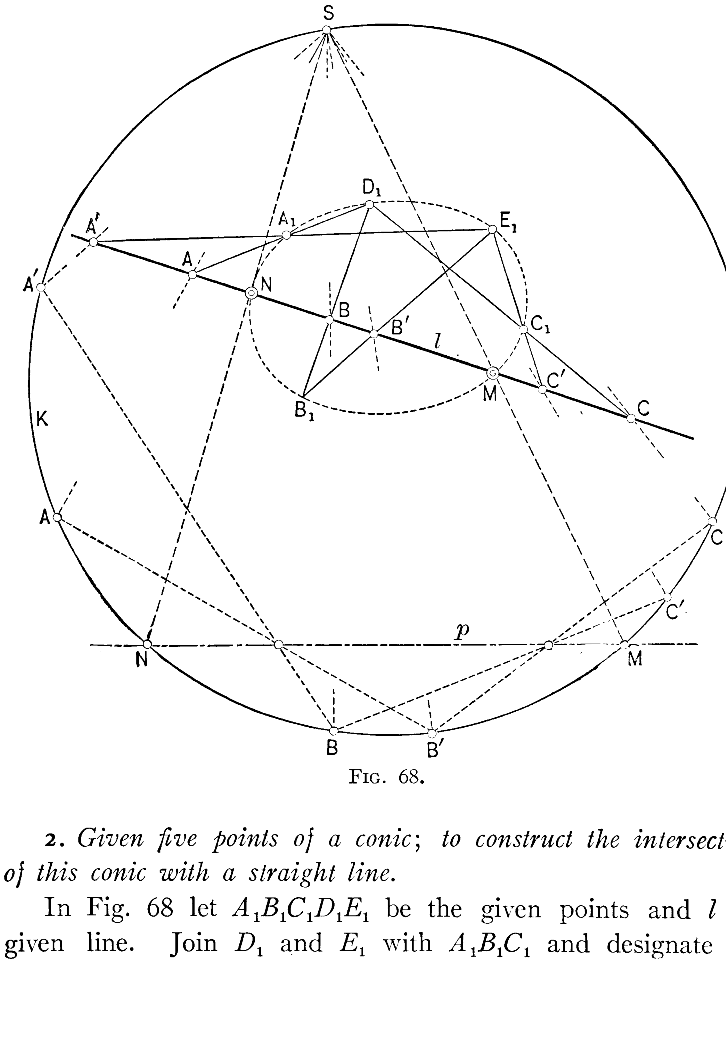


Fig. 68.

2. *Given five points of a conic; to construct the intersections of this conic with a straight line.*

In Fig. 68 let $A_1B_1C_1D_1E_1$ be the given points and l the given line. Join D_1 and E_1 with $A_1B_1C_1$ and designate the

corresponding intersections on l by ABC and $A'B'C'$. Draw an arbitrary circle K and join any of its points S with ABC and $A'B'C'$, and let these lines cut K in points which we shall also designate by ABC and $A'B'C'$, to simplify the designation. According to the foregoing results ABC and $A'B'C'$ determine two projective ranges on K. Construct the line p and let M and N be the intersections of p with K. Join S with M and N on K, and produce to their like-named intersections M and N on l. These will be the double-points of the projective

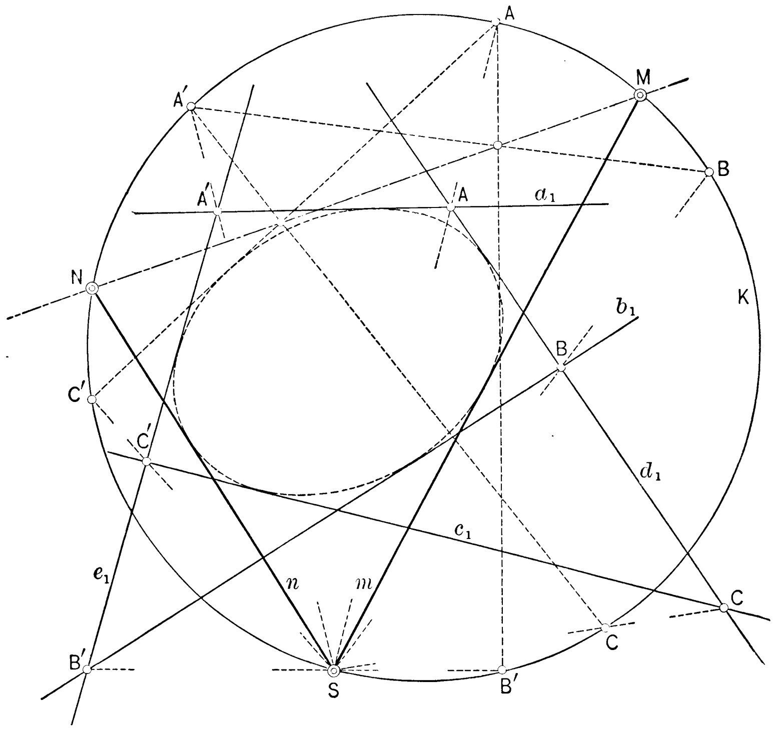


Fig. 69.

ranges $ABC\ldots$ and $A'B'C'$, or the points of intersection of l with the conic through $ABCDE$.

3. *Given five tangents of a conic; to construct the tangents of this conic through a given point.*

In Fig. 69 let a_1, b_1, c_1, d_1, e_1 be the given tangents and S the given point. Cut d_1 and e_1 with $a_1b_1c_1$ and designate the corresponding lines, joining these points with S, by $a\,b\,c$, $a'\,b'\,c'$. Draw an arbitrary circle K and let any tangent s of K cut $abca'b'c'$ in six points. From these draw tangents to K and designate them similarly by abc, $a'b'c'$. Construct the Brianchon point P of the circumscribed hexagon $abca'b'c'$ of K. From P draw the tangents m and n to K, cutting s in M and N. The lines joining S with M and N are the required tangents from S to the given conic.

This construction may be replaced by a simpler one. Join the points where a_1, b_1, c_1 cut d_1 and e_1 directly to the point S of an auxiliary circle passing through S, and designate the points of intersection with this circle by A, B, C, A', B', C'. Construct the Pascal line p of this hexagon. The lines joining S with the points of intersection of p with the auxiliary circle are the required tangents from S.

4. Poncelet's Problem.—*To construct a polygon whose vertices shall lie on given straight lines (each on each), and whose sides shall pass through given points (each through each).*

For the sake of simplicity we shall limit the problem to four straight lines a, b, c, d and four points A, B, C, D. The method of reasoning is not different in the general case. First make a trial construction by drawing through A a line a_1 cutting a in A_1, Fig. 70. From A_1 draw a line b_1 through B, cutting b in B_1; from B_1 a line c_1 through C, cutting c in C_1; from C_1 a line d_1 through D, cutting d in D_1; from D_1 a line a_1' through A cutting a in A_1'. If a_1 turns about A, then b_1, c_1, d_1, a_1' will turn about B, C, D, A in such a manner that we have for various positions the projective ranges

$$(A_1A_2A_3\ldots)=(B_1B_2B_3\ldots)=(C_1C_2C_3\ldots)=(D_1D_2D_3\ldots)=(A_1'A_2'A_3'\ldots).$$

Considering the projective ranges

$$(A_1A_2A_3\ldots)=(A_1'A_2'A_3'\ldots)$$

on a, it is clear that the lines a_M, a_N through A and the double-points of these ranges, M and N, coincide with the lines a_M', a_N' through D. These double-points determine, therefore, two solutions of the problem which may be real or imaginary. In

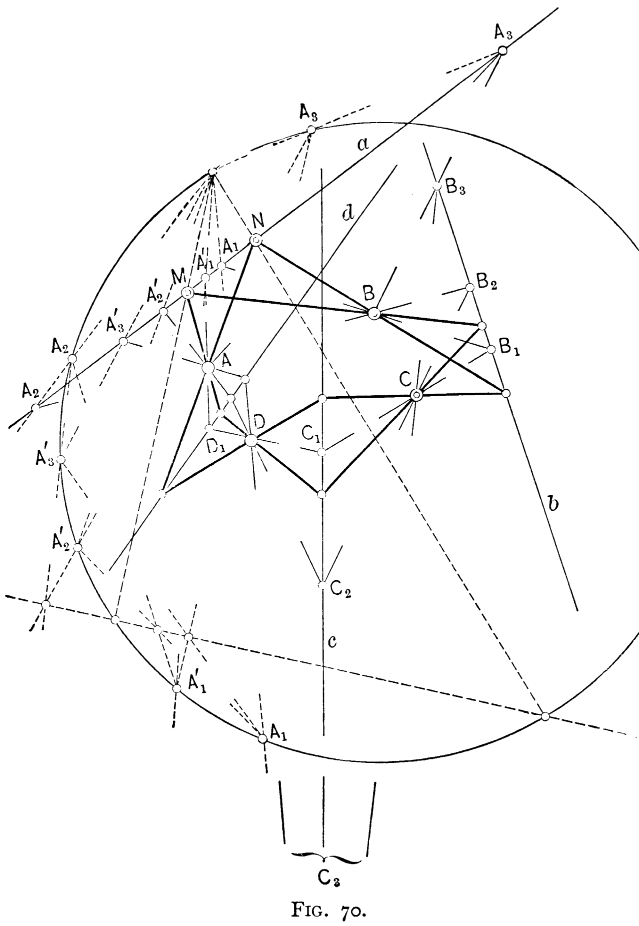


FIG. 70.

the figure the two real solutions of the problem are indicated by heavy lines.

Ex. 1. To inscribe in a given conic a polygon whose sides pass respectively through given, non-collinear, points.

Ex. 2. To circumscribe about a given circle a triangle whose vertices are on three given lines.

Ex. 3. Between two given straight lines to place a segment such that it shall subtend given angles at two given points.

Ex. 4. To construct a polygon whose sides shall pass respectively through given points, and all whose vertices except one shall lie respectively on given straight lines; and which shall be such that the angle included by the sides which meet in the last vertex is equal to a given angle. (Cremona.)

Let $A, B, C, \ldots, N$ be the given points and $a, b, c, \ldots, m$ the given lines. Through A and N draw a circle K which subtends the given angle over the chord AN. From any point of K draw a line through A, cutting a in A_1; from A_1 draw a line through B, cutting b in B_1, and so forth, until the line m is reached in a point M_1. Then through N and the same point on K draw a line cutting m in M_1'. Repeat this construction for two other points of K, thus giving on m the projective ranges $(A_1A_2A_3\ldots) = (A_1'A_2'A_3'\ldots)$. The double-points of these ranges make it possible to draw two polygons with the required properties. This problem may be solved in a different manner.

Through A draw any line a_1 cutting a in A_1, through A_1 and B a line b_1, and so forth, until the line m is reached in M_1. Through M_1 and N draw a line n_1 cutting a_1 in a point V. If now a_1 turns about A, then n_1 will turn projectively about N. Hence their point of intersection V will describe a conic K^* passing through A and N. The two other points of intersection of this conic with the circle K determine the two solutions of the problem. It is now possible that the conic K^* is itself a circle, but different from K. In this case there is no real solution. K^* may be identical with K, so that there are an infinite number of solutions.

Make the constructions as indicated.

§ 44. An Optical Problem.

1. The following problem stated by Cremona in his Elements of Projective Geometry, p. 199, is an application of Ex. 4 of the previous section:

A ray of light emanating from a given point A is reflected from n given straight lines in succession; to determine the original direction which the ray must have, in order that this may make with its direction after the last reflection a given angle.

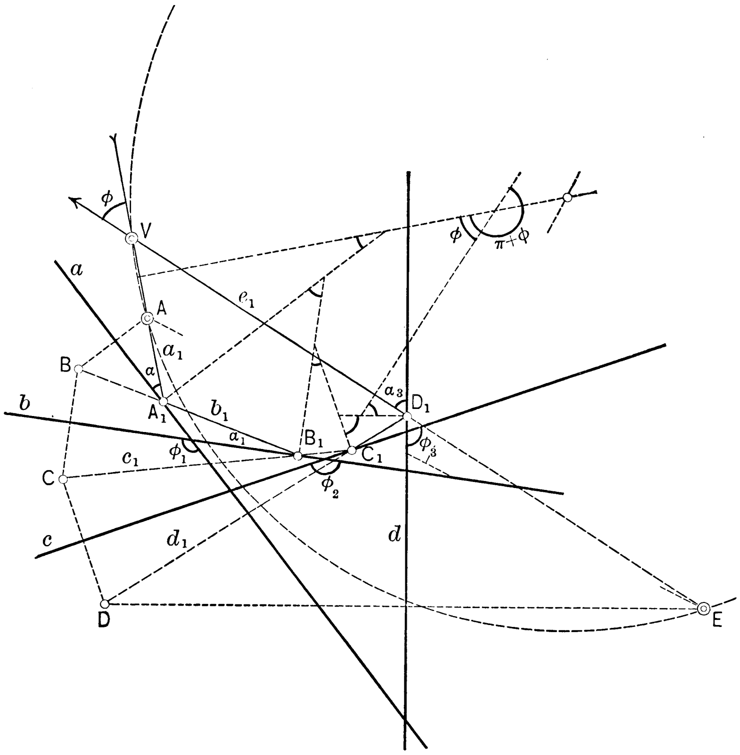


FIG. 71.

Designate in Fig. 74 the reflecting lines by $a, b, c, \ldots, n$. Through A draw any ray a_1 striking a at A_1. The reflected

ray b_1 passes through the point B which is symmetrical to A with respect to a. The ray b_1 strikes b at B_1, and its reflected ray c_1 passes through C, which is symmetrical to B with respect to b, and so forth. The ray o_1 reflected from the last line n at N_1 passes through O, which is symmetrical to N with respect to n. Let the rays a_1 and o_1 intersect at V_1. We have now a closed polygon $a_1b_1c_1 \ldots o_1$, whose sides pass through the fixed points $A, B, C, \ldots N, O$ and whose vertices except V lie on the fixed sides $a, b, c, \ldots, n$. Hence, when a_1 turns about A, V will describe a conic and the problem is reduced to the one explained in Ex. 4, § 43.

2. Cremona stops the discussion of the problem at this point. We shall now show that a further investigation is necessary. Let α be the angle of incidence of a_1 on a, α_1 the angle of incidence of b_1 on b, α_2 of c_1 on c, and so forth; $\phi_1, \phi_2, \phi_3 \ldots$ the angles which a and b, b and c, c and $d, \ldots$ include. The angles $\phi_1, \phi_2, \phi_3, \ldots$ between the different reflecting lines must be selected in such a manner that always

$$\alpha_i + \alpha_{i+1} + \phi_i = \pi.$$

From the figure we now derive the following series:

$$\begin{aligned}
\alpha_1 &= \alpha, \\
\alpha_2 &= \pi - \alpha - \phi_1, \\
\alpha_3 &= \alpha + \phi_1 - \phi_2, \\
\alpha_4 &= \pi - \alpha - \phi_1 + \phi_2 - \phi_3, \\
\alpha_5 &= \alpha + \phi_1 - \phi_2 + \phi_3 - \phi_4, \\
\alpha_6 &= \pi - \alpha - \phi_1 + \phi_2 - \phi_3 + \phi_4 - \phi_5, \\
&\;\;\vdots \\
\alpha_{2\mu} &= \pi - \alpha - \phi_1 + \phi_2 - \phi_3 + \ldots - \phi_{2\mu-1}, \\
\alpha_{2\mu+1} &= \alpha + \phi_1 - \phi_2 + \phi_3 - \ldots - \phi_{2\mu}.
\end{aligned}$$

If there are n reflecting lines, then the number of angles α is also n, and the number of angles ϕ is $n-1$. Erect perpendicu-

lars to a_1 at any of its points, to a at A_1, to b at B_1, and so forth, to n at any of its points. Then the first and the last perpendicular deviate from each other by the angle

$$\alpha+(n-2)\pi+\alpha_n-(\phi_1+\phi_2+\phi_3+\ \dots\ +\phi_{n-1}),$$

and consequently also the rays a_1 and o_1 by the angle ϕ:

$$\phi=\alpha+\alpha_n-(\phi_1+\phi_2+\phi_3+\ \dots\ +\phi_{n-1})+(n-2)\pi.$$

If n is odd, then $n-1$ is even; i.e., $n=2\mu+1$, and

$$\phi=\alpha+\alpha+\phi_1-\phi_2+\phi_3-\dots-\phi_{2\mu}-\phi_1-\phi_2-\phi_3-\dots-\phi_{2\mu}+(n-2)\pi,$$

or

$$\phi=(n-2)\pi+2\alpha-2(\phi_2+\phi_2+\phi_4+\dots+\phi_{2\mu});$$

i.e., the angle between the original incident ray of light and the final emanant ray depends upon the angles ϕ and the original angle of incidence α. Their point of intersection V describes a conic which is not a circle, and there are two positions of V, real or imaginary, for which the incident and reflected ray make a given angle.

If n is even, then $n-1$ is odd; i.e., $n=2\mu$ and

$$\phi=\alpha+\pi-\alpha-\phi_1+\phi_2-\phi_3+\dots-\phi_{2\mu-1}-\phi_1-\phi_2-\dots-\phi_{2\mu-1}+(n-2)\pi,$$

or

$$\phi=(n-1)\pi-2(\phi_1+\phi_3+\dots+\phi_{2\mu-1}).$$

Hence, in case of an even number of reflecting sides, V describes a circle and the angle ϕ is constant. Cremona's problem admits either of no solution, or of an infinite number of solutions. The angle ϕ does not depend upon the angles ϕ of even indices.

To sum up we may state Cremona's problem and its solution by the following proposition:

If rays of light emanate from a fixed source which, in succession, are reflected on n straight lines, then the last reflected rays cut the

corresponding original rays in points of a conic, which is not a circle, when n is odd, and in points of a circle when n is even. In

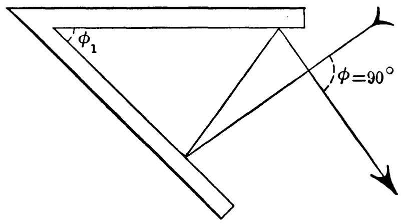


FIG. 72.

the first case there are two places on the conic at which the original and the final ray make a given angle. In the second case there are no such places on the circle, or else an infinite number. In this

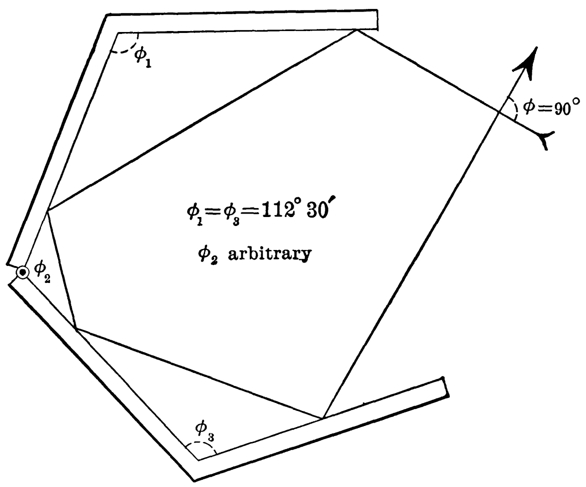


FIG. 73.

case the angle ϕ depends only upon the angles between succeeding reflecting lines whose orders in this succession are odd.

3. APPLICATIONS.—Let $n=2$, then $\phi=\pi-2\phi_1$.

To make $\phi=\frac{\pi}{2}$, we must choose $\phi_1=45°$. This case is illus-

trated in Fig. 72 and is practically applied in Bauernfeind's *Angle Mirror* or *Optical Square.*

For $n=4$

$$\phi=3\pi-2(\phi_1+\phi_3).$$

To make $\phi=\frac{\pi}{2}$, $\phi_1+\phi_3$ must be made equal to $\frac{5}{4}\pi$. Under this condition ϕ_1 and ϕ_3 may vary separately. The condition is also satisfied by taking $\phi_1=\phi_2=\frac{5}{8}\pi=112^\circ\ 30'$, and this is illustrated in Fig. 73.

CHAPTER IV.

PENCILS AND RANGES OF CONICS. THE STEINERIAN TRANSFORMATION. CUBICS.

§ 45. Pencils and Ranges of Conics.

1. *Involution of the Pencil* $u+\lambda u_1$[1] $=0$.

Let
$$u=ax^2+2bxy+cy^2+2dx+2ey+f=0,$$
$$u_1=a_1x^2+2b_1xy+c_1y^2+2d_1x+2e_1y+f_1=0$$

be the equations of two conics; then

$$(1)\qquad u+\lambda u_1=0$$

is the equation of a conic which passes through the four points of intersection of U and U_1. As a conic is determined by five points, any fifth point, different from one of the four points of intersection of U and U_1, determines the equation of the conic through the five points; i.e., λ. Conversely, every value of λ determines the equation of one of the conics of the system. Designating the points of intersection of U and U_1 by A, B, C, D, then for a variable λ,

$$u+\lambda u_1=0$$

represents all conics through the four fixed points A, B, C, D, and is called the equation of the *pencil of conics* through these points. Among these conics are three degenerate conics, con-

[1] See Joachimsthal, loc. cit., p. 183. By U, V, U_1, etc., we shall designate conics whose equations are $u=0$, $v=0$, $u_1=0$, etc.

The student is asked to draw a figure for this section.

isting of the three pairs of lines through A, B, C, D. To prove this we form the discriminant of (1), which is

$$(2) \qquad \begin{vmatrix} a+\lambda a_1 & b+\lambda b_1 & d+\lambda d_1 \\ b+\lambda b_1 & c+\lambda c_1 & e+\lambda e_1 \\ d+\lambda d_1 & e+\lambda e_1 & f+\lambda f_1 \end{vmatrix}$$

The vanishing of this expression is the condition for degenerate conics among the pencils. This gives a cubic equation in λ and consequently three values for λ; i.e., three degenerate conics through A, B, C, D, as was to be proved. One value of λ is always real, so that also in case of one or two imaginary pairs among A, B, C, D there is always a conic consisting of a real line-pair. In case of a double-root which is evidently real, the third root is also real; the conics U and U_1 have a contact of the first order. If the root is triple, U and U_1 have a contact of the second or third order.

In § 36 it was shown that the coordinates of a point $C(x, y)$ on the line joining the two points $A(x_1, y_1)$, $B(x_2, y_2)$ are

$$x=\frac{x_1-\lambda x_2}{1-\lambda}, \quad y=\frac{y_1-\lambda y_2}{1-\lambda}, \quad \text{where} \quad \lambda=\frac{AC}{BC}.$$

Assume now that A and B are on the conic given by (1), then C is on the conic U, if $\lambda=\dfrac{AC}{BC}$ is a root of

$$(3) \qquad u_1-2\lambda v+\lambda^2 u_2=0,$$

where u_1, v, u_2 have the same meaning as in formula (2), § 36. In a similar manner, C is on U_1, if λ satisfies

$$(4) \qquad u_1'-2\lambda v'+\lambda^2 u_2'=0,$$

where u_1', v', u_2' have the same meaning as in (3), except that a, b, c, ... are replaced by a_1, b_1, c_1, ... Designating the points

of intersections of AB produced with U by C, C'; and with U_1 by C_1, C_1', we have

$$\frac{AC}{BC}\cdot\frac{AC'}{BC'}=\frac{u_1}{u_2}; \quad \frac{AC_1}{BC_1}\cdot\frac{AC_1'}{BC_1'}=\frac{u_1'}{u_2'}.$$

The points A and B are on $u+\lambda u_1=0$; hence

$$u_1+\lambda u_1'=0, \quad u_2+\lambda u_2'=0,$$

and, eliminating λ,

$$\frac{u_1}{u_2}=\frac{u_1'}{u_2'}, \quad \text{or}$$

(5) $$\frac{AC}{BC}\cdot\frac{AC'}{BC'}=\frac{AC_1}{BC_1}\cdot\frac{AC_1'}{BC_1'}.$$

Giving λ all possible values and keeping the transversal, or C, C' and C_1, C_1', fixed, A, B is the pair of points in which the variable conic $u+\lambda u_1=0$ cuts this transversal. (5) may also be written

$$(ABCC_1)=(BAC'C_1');$$

i.e., the anharmonic ratio of any four points cut out by the fixed and variable conics on the transversal is equal to the anharmonic ratio of the four corresponding points. Furthermore, from (5) it is seen that interchanging A and B, two corresponding points, does not affect the relation. The system of points defined by (5) is therefore involutoric. Hence the theorem:

The conics of a pencil of conics cut any transversal in an involution of points. Every conic, including the degenerate conics, cuts out a pair of the involution. (Desargues.)

The double-points of the involution are evidently the points where two conics of the pencil touch the transversal. They may be real (including coincidence) or imaginary. The remark in

connection with problem 8, § 42, to construct a conic through four given points, tangent to a given line, is now clear.

COROLLARY.—*Any transversal cuts the three pairs of sides of a complete quadrilateral in three pairs of an involution.*

Ex. Prove directly that every transversal cuts a coaxial system of circles in an involution. By reciprocation we derive the theorem:

The pairs of tangents from any point to the conics of a range (conics inscribed in a quadrilateral) form an involutoric pencil.

COROLLARY.—*The lines joining any point with the three pairs of vertices of a complete quadrilateral form three pairs of an involutoric pencil.*

Ex. Prove directly that the tangents from any point to the range of circles inscribed to two straight lines form an involutoric pencil.

2. *A Special Case.*—Assume the four points A, B, C, D, through which the pencil of conics passes, as an orthogonal quadrilateral; i.e., $AB \perp CD$, $BC \perp AD$, $CA \perp BD$. *In this case the conics are all equilateral hyperbolas.* To prove this, note that the degenerate conics consist of three pairs of perpendicular lines, Fig. 74. The involution on the infinitely distant line is therefore rectangular and its pairs can only be cut out by conics whose infinite branches are rectangular; i.e., branches of equilateral hyperbolas.

Take now any equilateral hyperbola and on it any triangle ABC. Let D be the point of concurrence of the altitudes of ABC. Through $ABCD$ we can now pass an infinite number of equilateral hyperbolas, among which is necessarily the given hyperbola. Hence D is on this hyperbola, and we have the theorem:

The point of concurrence of the altitudes of any triangle inscribed in an equilateral hyperbola lies on this hyperbola.[1]

3. *Polars of a Pencil of Conics.*

---

[1] BRIANCHON et PONCELET in Gergonne's Annales, Vol. II, pp. 205–220. Also FIEDLER in *Vierteljahrsschrift d. Naturf.-Ges.*, Zürich, Vol. XXX, pp. 390–402.

From the explicit expression of the equation of a pencil of conics

(6) $$u + \lambda u_1 = 0$$

it is easily found that the equation of the polar of a point P may always be put in the form

(7) $$p + \lambda p_1 = 0,$$

where $p = 0$, $p_1 = 0$ are the equations of the polars of P with respect to the conics U and U_1. From this it follows that the

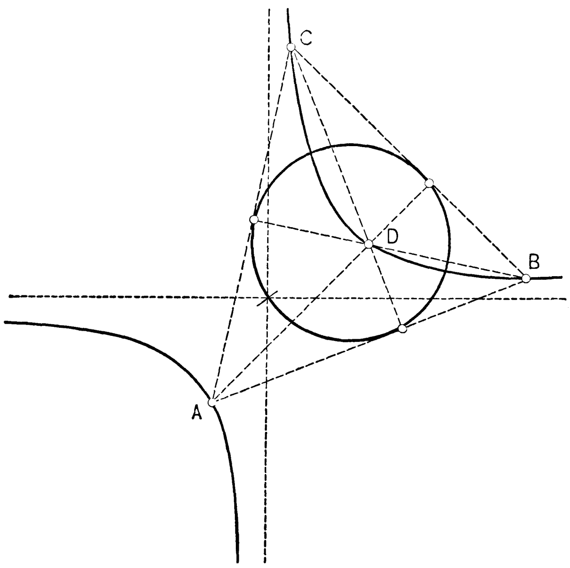


FIG. 74.

polar of any point P with respect to a conic of the pencil always passes through the point of intersection P' of the polars of P with respect to U and U_1. Hence the theorem:

All polars of a point with respect to the conics of a pencil are concurrent.

If the point P with the coordinates x_1, y_1 describes the straight

line $\alpha x_1+\beta y_1+\gamma=0$, then for the point $P'=(x_1'y_1')$ we have the three conditions

$$
\begin{array}{lllll}
(ax_1'+by_1'+d)x_1+ & (bx_1'+cy_1'+e)y_1+ & dx_1'+ey_1'+f & & =0, \\
(a_1x_1'+b_1y_1'+d_1)x_1+ & (b_1x_1'+c_1y_1'+e_1')y_1+ & d_1x_1'+e_1y_1'+f_1 & & =0, \\
\qquad \alpha x_1+ & \qquad \beta y_1+ & \qquad \gamma & & =0,
\end{array}
$$

which are consistent only when

$$
(8) \qquad \begin{vmatrix} ax_1'+by_1'+d & bx_1+cy_1'+e & dx_1'+ey_1'+f \\ a_1x_1'+b_1y_1'+d_1 & b_1x_1'+c_1y_1'+e_1 & d_1x_1'+e_1y_1'+f_1 \\ \alpha & \beta & \gamma \end{vmatrix}=0.
$$

This gives a quadratic equation between x_1', y_1', the coordinates of P'; hence the theorem:

If a point P describes a straight line, then the point of concurrence P' of all polars with respect to the conics of a pencil describes a conic.

Designating $ax_1'+by_1'+d$, $bx_1'+cy_1'+e$, $dx_1'+ey_1'+f$ by r, s, t, and in a similar manner by r_1, s_1, t_1 the same expressions, with a, b, e, ... replaced by a_1, b_1, c_1, the polar of a point (x_1', y_1') with respect to the pencil of conics $u+\lambda u_1=0$ has the form

$$(r+\lambda r_1)x_1+(s+\lambda s_1)y_1+t+\lambda t_1=0.$$

This equation will be identical with that of the given line g,

$$\alpha x_1+\beta y_1+\gamma=0, \quad \text{if}$$

$$
(9) \qquad \frac{r+\lambda r_1}{\alpha}=\frac{s+\lambda s_1}{\beta}=\frac{t+\lambda t_1}{\gamma}.
$$

Hence the pole (x_1', y_1') of g for any conic of the pencil must satisfy (9). For every value of λ a definite value of (x_1', y_1') is obtained, which therefore describes a certain locus. To find

its equation we must eliminate λ from (9), which gives the equation

$$(10)\qquad \alpha(st_1-s_1t)+\beta(r_1t-rt_1)+\gamma(rs_1-r_1s)=0.$$

But this is identical with (8). Hence the theorem:

The locus of the poles of a straight line with respect to all conics of a pencil is a conic which is identical with the conic of concurrent polars of all points of the given line with respect to the same pencil.

Of particular interest is the case when P describes the infinitely distant line; i.e., when $\frac{y_1}{x_1}=\mu$ is a variable (finite) and $x_1=\infty$. In this case

$$(11)\qquad \begin{cases} p\equiv ax_1'+by_1'+d+\mu(bx_1'+cy_1'+e)=0, \\ p_1\equiv a_1x_1'+b_1y_1'+d_1+\mu(b_1x_1'+c_1y_1'+e_1)=0. \end{cases}$$

Eliminating μ, the equation of the conic which P' describes becomes

$$(12)\qquad (ax_1'+by_1'+d)(b_1x_1'+c_1y_1'+e_1) \\ -(a_1x_1'+b_1y_1'+d_1)(bx_1'+cy_1'+e)=0.$$

If (x_1, y_1) describes the infinitely distant line, then its pole with respect to a certain conic of $u+\lambda u_1=0$ must satisfy $p+\lambda p_1=0$ for all values of μ, which can only be true when equation (12) is satisfied. But the poles of the infinite line are the middle points of the conics of the pencil. *The centers of a pencil of conics lie on a conic whose equation is given by* (12).

The three diagonal points of the fundamental quadrilateral, being the centers of the three degenerate conics, belong to this locus. If P is taken as the infinitely distant point of the line joining two points, say A and B, of the fundamental quadrilateral, then P' is the middle point of AB, since all polars of P with respect to the conics of the pencil pass through this point. The locus (12) passes, therefore, also through the middle points

of AB, BC, CD, DA, BD, CA. In case of an orthogonal quadruple, as it was described above, under (2), the locus (12) becomes a circle circumscribed to the foot-points of the altitudes of the triangle ABC, which bisects the sides and the segments DA, DB, DC of the altitudes, Fig. 74. This circle is otherwise called the *Feuerbach circle* [1] of the triangle. We have therefore the theorem:

The locus of the centers of all equilateral hyperbolas circumscribed to an orthogonal quadrilateral $ABCD$ (D = point of concurrence of altitudes) is the Feuerbach circle of the triangle ABC.

4. *Poles of a Range of Conics.*

A range of conics consists of all conics inscribed to a quadrilateral (imaginary elements included), or is the reciprocal of a pencil of conics $u+\lambda u_1=0$ with respect to a given conic K. From this property we derive immediately the theorems:

All poles of a straight line with respect to the conics of a range are collinear.

If a straight line p turns about a fixed point, then the line of collinearity p' of all its poles with respect to the conics of a range envelops a conic.

Let M be the center of K and designate by V a conic of the pencil $u+\lambda u_1=0$, and by v the polar of M with respect to V. On reciprocation with respect to K, V is transformed into a conic V'; M, the pole of v, is transformed to infinity; consequently the polar v is transformed into the center of the transformed conic V'. As the polars of M with respect to all conics V are concurrent, it follows that their reciprocal poles are collinear.

Hence the theorem:

The centers of the conics of a range are collinear.

Among the conics of the range there are three degenerate ones, consisting of the three pairs of points in which the sides of the fundamental quadrilateral intersect each other. The

[1] Concerning this circle see CAJORI'S *History of Elementary Mathematics*, pp. 259, 260; also KÖTTER: *Die Entwickelung der synthetischen Geometrie*, Vol. I, pp. 35–38.

middle points of these three pairs evidently belong to the above locus. This may be stated in the corollary:

The middle points of the three diagonals of a complete quadrilateral are collinear.

Designating the three pairs of points by A_1A_2, B_1B_2, C_1C_2, two circles over A_1A_2 and B_1B_2 as diameters intersect at two points (real or imaginary) F_1, F_2. Joining F_1 to A_1, A_2 and B_1, B_2, the involution of the tangents from F_1 to all conics of the range is determined, and as two of these pairs, F_1A_1, F_1A_2 and F_1B_1, F_1B_2, are rectangular, all other pairs are rectangular; i.e., $F_1C_1 \perp F_1C_2$, and the circle over C_1C_2 as a diameter is coaxial with the first two circles.[1] Constructing all circles from the points of which rectangular pairs of tangents may be drawn to the conics, it follows from the last remarks that all these circles form a coaxial system. We state these facts once more in the theorem:

The circles from whose points pairs of perpendicular tangents may be drawn to the conics of a range, each for each, form a coaxial system.

Ex. 1. If through the vertices of two angles, whose sides intersect each other in the points A, B, C, D, two parallel lines are drawn, then the harmonic lines of each of these parallel lines with respect to the sides of the corresponding angles intersect each other in a point which describes the conic of the middle points of all conics through A, B, C, D, when the direction of the two parallel lines changes.

§ 46. Products of Pencils and Ranges of Conics.

1. The pencils and ranges of conics may be related to each other by requiring that two conics shall correspond to each other if their equations are determined by one and the same parameter λ. Thus, for a certain value of λ,

$$(1) \qquad \begin{cases} u+\lambda u_1 = 0, \\ v+\lambda v_1 = 0 \end{cases}$$

[1] The student is asked to make the foregoing construction.

represent two corresponding conics. The two pencils (1) are said to be projective. Two corresponding conics intersect each other in four points (including imaginary points). We obtain the locus of all these points by eliminating λ from equations (1), which gives

$$uv_1 - u_1v = 0, \tag{2}$$

an equation of the fourth degree in x and y. Hence the theorem:

Two projective pencils of conics produce a curve of the fourth order.

As the equation of a curve of the fourth order depends upon twelve constants and as (2) contains twenty constants, it is evidently always possible to state the converse; i.e.,

Every curve of the fourth order may be considered as the product of two projective pencils of conics.

The curve as represented by (2) passes through the intersections of U and U_1, V and V_1, U and V, U_1 and V_1.

Reciprocally we have the theorems:

Two projective ranges of conics produce a curve of the fourth class.

Every curve of the fourth class may be considered as the product of two projective ranges of conics.

2. In analogy to (1) a pencil of conics and a pencil of rays are projective if their equations may be written in the respective forms

$$\begin{cases} u + \lambda u_1 = 0, \\ l + \lambda l_1 \ = 0. \end{cases} \tag{3}$$

For every λ we have a conic and a ray corresponding to each other in this projectivity, and the two intersect each other in two points. The locus of these points is obtained by eliminating λ from equations (3); so that its equation is

$$ul_1 - u_1l = 0, \tag{4}$$

and is of the third degree. It is satisfied for $l=0$, $l_1=0$; $u=0$, $u_1=0$; $u=0$, $l=0$; $u_1=0$, $l_1=0$. Hence the theorem:

The product of a pencil of conics and a projective pencil of rays is a curve of the third order which passes through the vertex of the pencil of rays and through the four fundamental points of the pencil of conics.

As the equation of a cubic depends upon nine constants and as (4) contains fourteen constants, it is always possible to write the equation of any cubic in the form of (4). Hence the theorem:

Every cubic may be considered as the product of a pencil of conics and a projective pencil of rays.

Reciprocally:

The product of a range of conics and a range of points is a curve of the third class which is inscribed to the fundamental quadrilateral of the range of conics and which touches the range of points.

Conversely, *every curve of the third class may be considered as such a product.*

3. In § 45, 3, it was shown that the polars of a point P with respect to a pencil of conics $u+\lambda u_1=0$ are concurrent at a point P', and that when P describes a straight line, P' describes a conic. In general to a point P corresponds one and only one point P'. Let the straight line described by P be g and the corresponding conic described by P' be G, and designate the points where g cuts G by X and X', Fig. 75. The relation between P and P' is involutoric; i.e., all polars of P' with respect to the pencil pass through P.

To the point X on g corresponds a point on G, to X on G corresponds a point on g; but to X only one point corresponds in the correspondence between P and P'; hence the point corresponding to X is X'. Conversely, to X' corresponds X. The pencil of conics cuts g in an involution of points. Let M and N be the double-points of this involution, V_M and V_N the conics of the pencil touching g at M and N. Then, the polars of M with respect to V_M and V_N are g and the polar passing through N. Hence, in the correspondence of P and P', to M corresponds

the point N, and, conversely, to N corresponds M. There are only two points on g with this property, the points X and X' where g cuts the conic G. Consequently M and N coincide with X and X'. We may state this result in the theorem:

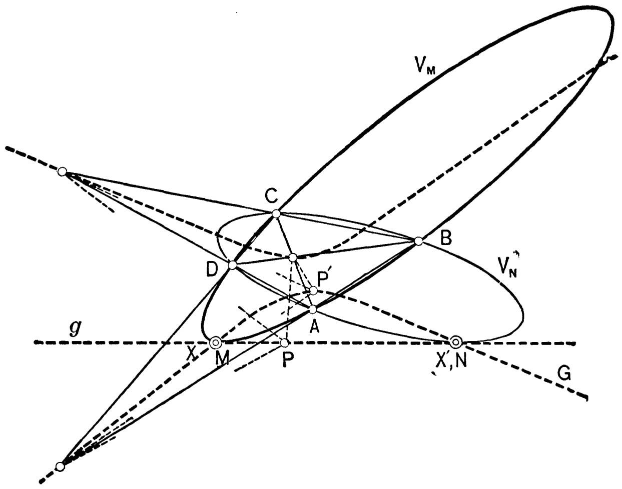


FIG. 75.

In the correspondence of P and P' to a straight line g corresponds a conic G. The points on g whose corresponding points are on g itself are the points of intersection X and X' of g with G. These same points are also the double-points of the involution of points which the pencil of conics cuts out on g.

According to the theorem that G is also the locus of the poles of g with respect to the conics of the pencil, the points X and X' on G are poles of g, and as these coincide with g it follows that g touches two conics of the pencil at X and X'; in other words, X and X' are the double-points of the involution cut out on g by the pencil of conics, as has been established above. The theorem therefore also holds for an imaginary pair of corresponding points X, X'.

4. Consider now the straight lines of a pencil:

(5) $$(\alpha+\mu\alpha_1)x+(\beta+\mu\beta_1)y+\gamma+\mu\gamma_1=0.$$

For a definite value μ we have a definite ray of the pencil. According to (8) in § 45, 3, when P describes the line (5), P' describes the conic

$$(6)\qquad \begin{vmatrix} ax_1'+by_1'+d & bx_1'+cy_1'+e & dx_1'+ey_1'+f \\ a_1x_1'+b_1y_1'+d_1 & b_1x_1'+c_1y_1'+e_1 & d_1x_1'+e_1y_1'+f_1 \\ \alpha+\mu\alpha_1 & \beta+\mu\beta_1 & \gamma+\mu\gamma_1 \end{vmatrix}=0,$$

which may also be written in the form

$$(7)\qquad \begin{vmatrix} ax_1'+by_1'+d & bx_1'+cy_1'+e & dx_1'+ey_1'+f \\ a_1x_1'+b_1y_1'+d_1 & b_1x_1'+c_1y_1'+e_1 & d_1x_1'+e_1y_1'+f_1 \\ \alpha & \beta & \gamma \end{vmatrix}$$

$$+\mu \begin{vmatrix} ax_1'+by_1'+d & bx_1'+cy_1'+e & dx_1'+ey_1'+f \\ a_1x_1'+b_1y_1'+d_1 & b_1x_1'+c_1y_1'+e_1 & d_1x_1'+e_1y_1'+f_1 \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}=0.$$

Designating $\alpha x+\beta y+\gamma$ and $\alpha_1 x+\beta_1 y+\gamma_1$ by g and g_1 and the corresponding conics by $G=0$ and $G_1=0$, then to the pencil $g+\mu g_1=0$ corresponds the projective pencil of conics $G+\mu G_1=0$. The product of the two pencils is therefore a curve of the third order with the equation

$$(8)\qquad Gg_1-G_1g=0.$$

In the transformation of P into P', to a pencil of rays corresponds a pencil of conics projective to the pencil of rays. The product of the two pencils is a curve of the third order. This curve may also be considered as the locus of those points on the rays of a pencil whose corresponding points are on the same rays, each for each.

Ex. 1. Establish the equation of a coaxial system of circles. Prove the propositions of this section directly in this special case.

Ex. 2. Prove that the pencil of rays joining any point to the centers of a coaxial system of circles is projective to this system. Establish the equation of the curve produced by the two pencils.

Ex. 3. Show in what manner a system of confocal conics may be considered as a range of conics.

Ex. 4. What is the fundamental quadrilateral in case of two conics $u=0$, $u_1=0$, having a double contact?

§ 47. The Steinerian Transformation.

1. In the foregoing sections we have shown that the polars of any point P with respect to a pencil of conics are concurrent at a point P'. For the construction and clear understanding of this transformation it is of great advantage to consider in particular the degenerate conics of the pencil through the quadrangle which shall be designated by $A_1A_2A_3A_4$, and its diagonal points by B_1, B_2, B_3. The pairs of lines A_1A_2, B_3A_3; A_2A_3, B_1A_1; A_3A_1, B_2A_2 are the degenerate conics of the pencil. To find P' when P is given, join P to B_1, B_2, B_3, Fig. 76, and construct the fourth harmonic rays to PB_1, PB_2, PB_3 with respect to the corresponding pairs of lines through B_1, B_2, B_3. The three harmonic rays intersect each other at P'. From this simple geometric construction it is now easy to study the correspondence of P and P' for any particular positions. At every point B, say B_3, the lines A_1A_2, B_3A_3, B_3B_1, B_3B_2 form a harmonic pencil. To the points B correspond, therefore, all points of their opposite sides of the triangle $B_1B_2B_3$. The points A_1, A_2, A_3, A_4 are invariant, since the fourth harmonic rays pass through the points themselves. To a point on any line joining two of the fundamental points, say A_1A_3, corresponds the fourth harmonic point to the pair A_1A_3. All other points are in uniform correspondence.

We have seen that to a straight line corresponds a conic. As a straight line cuts each of the sides B_1B_2, B_2B_3, B_3B_1, and as to these sides correspond the opposite points B_3, B_1, B_2, it follows that said conic passes through the points B_1, B_2, B_3. To the straight lines of the plane corresponds the *net of conics* through

[1] See STEINER'S collected works, Vol. I, pp. 407–421, and M. DISTELI: *Die Metrik der circularen Curven dritter Ordnung im Zusammenhang mit geometrischen Lehrsätzen Jakob Steiners.* Also PONCELET: *Traité*, 1 ed. 1822, p. 198.

$B_1B_2B_3$. Taking a pencil of rays through P, its corresponding pencil of conics passes through P' and B_1, B_2, B_3. The curve of the third order produced by these two pencils passes, therefore, according to § 46, (4), through P and P', B_1, B_2, B_3.

On every ray through P there are two corresponding points X and X' of the cubic. Consequently, connecting P to A_1, A_2,

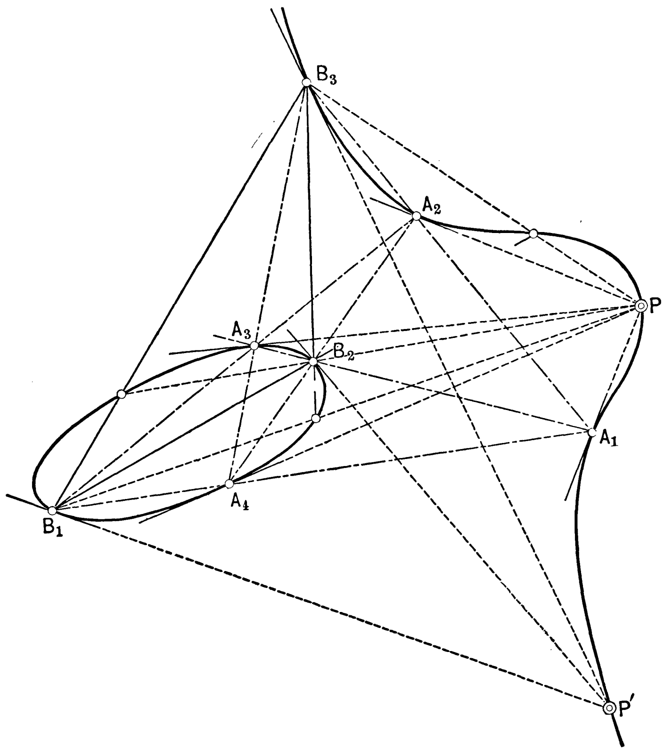


FIG. 76.

A_3, A_4, the corresponding points on these four rays coincide, each for each, with A_1, A_2, A_3, A_4, so that these points are on the cubic and PA_1, PA_2, PA_3, PA_4 the tangents at these points. On the rays PB_1, PB_2, PB_3 the points which correspond to B_1, B_2, B_3 are the points of intersection B_1', B_2', B_3' of these rays with the sides B_2B_3, B_3B_1, B_1B_2, respectively. The points B_1', B_2', B_3' are therefore also on the cubic. Hence the theorem:

In the Steinerian transformation to every pencil of rays corresponds a projective pencil of conics through the diagonal points of the fundamental quadrilateral. The product of the two pencils is a curve of the third order through the vertex of the pencil of rays and its corresponding point and through the vertices and diagonal points of the fundamental quadrangle. Thus to every point of the plane may be associated a certain curve of the third order in the Steinerian transformation. All these ∞^2 *cubics pass through seven fixed points.*

Without proceeding to the Steinerian transformation of conics, cubics, etc., we shall immediately take the general case of a curve of the nth order, C_n. To determine in how many points any straight line g cuts C_n, notice that the conic G corresponding to g cuts C_n in $2n$ points. Hence, when the whole configuration is transformed, G with its $2n$ intersections on C_n is transformed into $2n$ intersections of g with the transformed C_n. Hence the theorem:

In a Steinerian transformation a curve of the nth order is generally transformed into a curve of order 2n.

2. *Analytical Expression for a Steinerian Transformation.*

Nothing will be lost in the general result if we assume that the points A_1, A_2, A_3 form an equilateral triangle and that A_4 be its center, since by a collineation this orthogonal quadrangle may be transformed into any other quadrangle. Let A_4 coincide with the origin, and A_1 with the X-axis, Fig. 77, and $A_1A_4=A_2A_4=A_3A_4=1$. The foot-points of the perpendiculars of the triangle are B_1, B_2, B_3. Consider first the degenerate conic consisting of the lines A_1A_2 and A_3B_3 with the equations

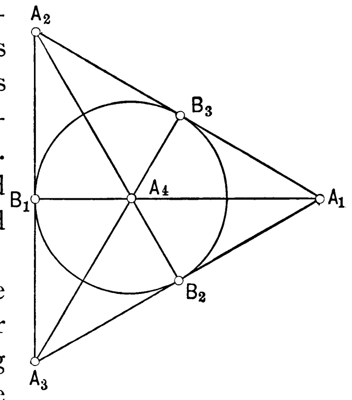


FIG. 77.

$$\xi+\sqrt{3}\cdot\eta-1=0,$$
$$\sqrt{3}\xi-\eta=0,$$

so that the equation of the degenerate conic is

$$(1)\qquad \sqrt{3}\cdot\xi^2-\sqrt{3}\cdot\eta^2+2\xi\eta-\sqrt{3}\cdot\xi+\eta=0.$$

Similarly the equation of the degenerate conic represented by A_1A_3 and A_2B_2 is

$$(2)\qquad \sqrt{3}\cdot\xi^2-\sqrt{3}\cdot\eta^2-2\xi y-\sqrt{3}\cdot\xi-\eta=0.$$

The equations of the polars of the point P (x, y) with respect to these conics are

$$(3)\qquad (x\sqrt{3}+y-\tfrac{1}{2}\sqrt{3})\xi+(x-y\sqrt{3}+\tfrac{1}{2})\eta-\frac{x}{2}\sqrt{3}+\frac{y}{2}=0,$$

$$(4)\qquad (x\sqrt{3}-y-\tfrac{1}{2}\sqrt{3})\xi-(x+y\sqrt{3}+\tfrac{1}{2})\eta-\frac{x}{2}\sqrt{3}-\frac{y}{2}=0.$$

The common solutions of (3) and (4) are the coordinates x', y' of the point P corresponding to P in the Steinerian transformation:

$$(5)\qquad \begin{cases} x'=\dfrac{2(x^2-y^2)+x}{4(x^2+y^2)-1}, \\[2ex] y'=\dfrac{y-4xy}{4(x^2+y^2)-1}. \end{cases}$$

Solving these equations with respect to x and y we obtain

$$(6)\qquad \begin{cases} x=\dfrac{2(x'^2-y'^2)+x'}{4(x'^2+y'^2)-1}, \\[2ex] y=\dfrac{y'-4x'y'}{4(x'^2+y'^2)-1}, \end{cases}$$

which shows that the transformation is involutoric.

To the line at infinity, $x=\infty$, $y=\infty$, $\dfrac{y}{x}=$arbitr., corresponds the circle

$$(7)\qquad x'^2+y'^2=\tfrac{1}{4}.$$

Ex. 1. The centers of the conics circumscribed to a quadrangle $A_1A_2A_3A_4$ lie on a conic K, which bisects the distances between these points in six points. These form three parallelograms having the same center, which is the center of the conic cutting out these points.

Ex. 2. According as a straight g cuts K, in two real or two imaginary points, or touches it, the corresponding conic in the Steinerian transformation will be an hyperbola, an ellipse, or a parabola.

Ex. 3. Prove by formulas (5) that in a Steinerian transformation a C_n is transformed into a C_{2n}. In particular a straight line is transformed into a conic.

Ex. 4. Prove that in the Steinerian transformation $A_1A_2A_3A_4$ are invariant points and that to the sides B_1B_2, B_2B_3, B_3B_1 correspond the points B_3, B_1, B_2, by using formulas (5).

§ 48. Curves of the Third Order.

1. In the Steinerian transformation, with every point of the plane is associated a certain cubic. As in the previous section assume as conics determining the fundamental quadrangle or quadruple the degenerate conics.

$$u \equiv \sqrt{3}\cdot x^2 + 2xy - \sqrt{3}\cdot y^2 - \sqrt{3}\cdot x + y = 0, \tag{1}$$

$$u_1 \equiv \sqrt{3}\cdot x^2 - 2xy - \sqrt{3}\cdot y^2 - \sqrt{3}\cdot x - y = 0. \tag{2}$$

To find the cubic associated with the point (x', y'), take as lines g and g_1 in formula 8, § 46,

$$g \equiv x - x' = 0, \tag{3}$$

$$g_1 \equiv y - y' = 0. \tag{4}$$

According to (5), § 47, to these lines correspond in the Steinerian transformation the conics

$$G \equiv 2(x^2 - y^2) + x - x'\{4(x^2 + y^2) - 1\} = 0, \tag{5}$$

$$G_1 \equiv \quad y - 4xy \quad - y'\{4(x^2 + y^2) - 1\} = 0. \tag{6}$$

The equation of the cubic associated with the point (x', y') is $Gg_1 - G_1g = 0$, or

$$\begin{aligned} &(y-y')\{2(x^2-y^2)+x-x'[4(x^2+y^2)-1]\} \\ -&(x-x')\{y-4xy \qquad -y'[4(x^2+y^2)-1]\}=0, \end{aligned}$$

or

$$(7) \qquad \left\{ \begin{aligned} &(y-y')\left(\frac{2(x^2-y^2)+x}{4(x^2+y^2)-1}-x'\right) \\ -&(x-x')\left(\frac{y-4xy}{4(x^2+y^2)-1}-y'\right)=0. \end{aligned} \right.$$

From the form of this equation it is apparent that a Steinerian transformation does not change the equation. Hence the theorem:

The net of cubics through a quadruple and its diagonal points is invariant in the corresponding Steinerian transformation.

This is also geometrically evident. In the construction of the curve, Fig. 76, eleven points are obtained through which the cubic passes and which, as a group, are invariant in the Steinerian transformation.

For the points $x'=\infty$, $y'=\infty$, $\dfrac{y'}{x'}=\kappa$, (7) reduces to

$$(8) \qquad y+\kappa x+\frac{y-4xy}{4(x^2+y^2)-1}-\kappa\frac{2(x^2-y^2)+x}{4(x^2+y^2)-1}=0.$$

Also in this case the cubic is the locus of the double-points of the involutions cut out on the pencil of parallel rays through the infinite point $\left(\dfrac{y'}{x'}=\kappa\right)$ by the pencil of conics through the fundamental quadruple. The line at infinity belongs also to the pencil of parallel rays, and as the involution on it is rectangular it follows that the double points are the circular points. Hence (8) *represents a pencil of bicircular cubics.*

As has been seen already, the tangents to the cubic at the points $A_1A_2A_3A_4$ pass through the point P. We shall now prove that the tangents at B_1, B_2, B_3 pass through P'. For this purpose

draw a ray through P cutting the cubic in two points U and V, of which U shall be close to B_2, Fig. 78. To this ray corresponds

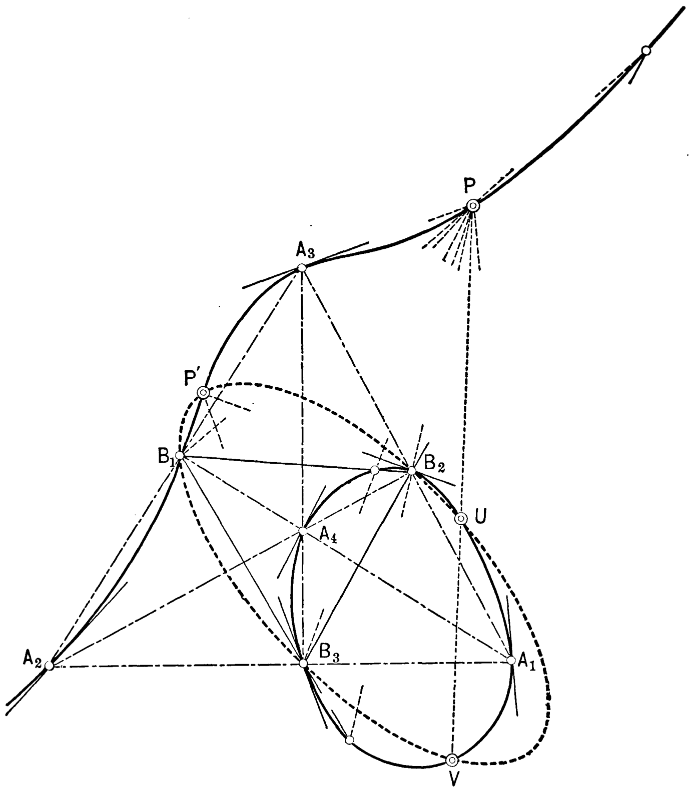


FIG. 78.

in the Steinerian transformation a conic through B_1, B_2, B_3, U, V, and P'. As the ray through P turns in such a manner that U approaches B_2 as a limit, the corresponding conic will approach the degenerated conic, consisting of the ray $P'B_2$ and the side B_1B_3 as a limit. Hence, when the ray passes through B_2, the corresponding ray through P' will be a tangent to the cubic at B_2. A similar result is obtained for the points B_1 and B_2, which proves the proposition.

2. In what follows it will be assumed that the cubic is a circular curve; i.e., that the point P is infinitely distant. Designating this infinitely distant point by B and its corresponding point by C, the tangents at the A's are parallel to the direction of B, and the tangents at the B's pass through C, Fig. 79. The ray through C parallel to the direction of B is the asymptote of the curve. Hence the tangents at the points B, B_1, B_2, B_3 meet in the point C of the same curve. Four points on the cubic with this property are called a *Steinerian quadruple* of the cubic.

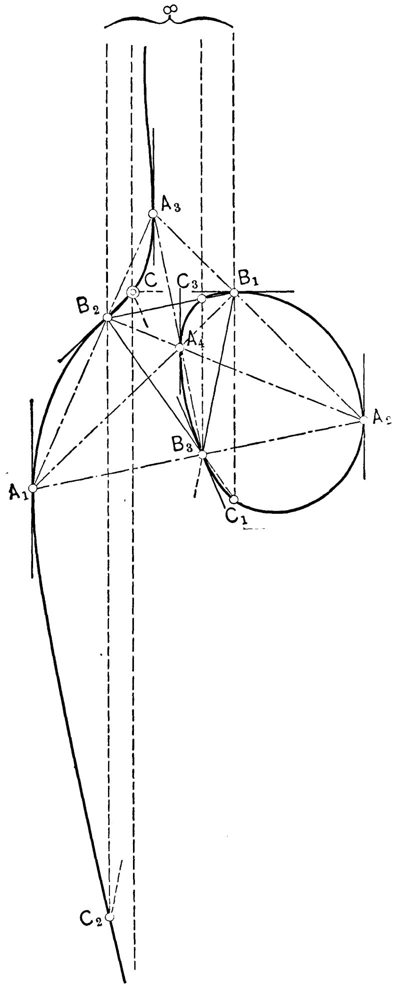


FIG. 79.

Thus $A_1A_2A_3A_4$, $BB_1B_2B_3$ are such quadruples. According to previous results, the rays BB_1, BB_2, BB_3 cut the opposite sides B_2B_3, B_3B_1, B_1B_2 in three more points, C_1, C_2, C_3, of the cubic. But this is equivalent with considering $BB_1B_2B_3$ as a fundamental quadrangle in a new Steinerian transformation with C_1, C_2, C_3 as the diagonal points, and C as the original point associated with the cubic. That the cubic associated with C in this new transformation is identical with the original cubic follows from the following consideration: The points B being points of tangency count for eight given points. Furthermore, the four C's lie on the original curve, so that the new curve has

at least twelve points in common with the original cubic, and is consequently identical with it.

In this new Steinerian transformation construct the point D corresponding to C. Then take the new quadruple $CC_1C_2C_3$ and construct the associated cubic in the Steinerian transformation belonging to this quadruple. The new cubic is identical with the original cubic, as can easily be proved. The tangents at C, C_1, C_2, C_3 all pass through D. For the quadruple $CC_1C_2C_3$ construct the diagonal points D_1, D_2, D_3. These together with D form a new quadruple, whose tangents pass through E, the point corresponding to D in the transformation associated with the quadruple $C_1C_2C_3C_4$. Continuing this construction,[1] we may obtain any number of points of the cubic arranged in quadruples. The points B, C, D, E . . . have the property that the tangent at one of these points always passes through the previous point.

3. The general equation of a cubic may be written

$$\begin{aligned} (1) \qquad & Ax^3+Bx^2y+Cxy^2+Dx^3+ \\ & ax^2+2bxy+cy^2+2dx+2ey+f=0. \end{aligned}$$

The problem arises, what connection exists between the fundamental quadruple with which the cubic is associated and the shape or equation of the cubic. In the above discussion the quadruple was assumed as real and the cubic consisted of a serpentine (infinite branch) and an oval. By certain collineations this curve may be transformed into various other curves which may be characterized with respect to their behavior at infinity. The serpentine or oval will be called elliptic, hyperbolic, or parabolic, according as they have two imaginary, two real, or two coincident points at infinity. Designating by r the counter-axis which in a collineation is transformed to infinity, and by S and O the serpentine and oval of the cubic, then the transformed curves resulting from various positions of r are as given in the following table:

[1] For the sake of simplicity, in the figure only the quadruple $CC_1C_2C_3$ has been constructed.

r	Original Curve.		Resulting Curve.	
	S	O	S	O
cutting	in 3 points	....	hyperbolic	elliptic
tangent	in 1 point	....	parabolic	elliptic
cutting	....	in 2 points	elliptic	hyperbolic
tangent	....	in 1 point	elliptic	parabolic

From these possible collineations it is seen that a cubic with two branches, serpentine and oval, by any collineation is transformed into a cubic with two branches. The geometrical discussion of this section therefore does not cover all cases as represented by the general equation of the cubic. For this purpose it is necessary to classify the cubics from the general equation, or the fundamental quadruple, by introducing coincident and imaginary elements. We shall do both. As the analytical discussion is briefer, we shall take this up first and discuss the geometrical aspect later on. To equation (1) apply the general projective transformation or collineation of the xy-plane as given in § 19. This collineation depends upon eight parameters. After the transformation, clearing of fractions, collection of equal terms in x and y, (1) assumes the form

$$(2)\qquad \left\{\begin{array}{l} A_1x^3+B_1x^2y+C_1xy^2+D_1y^3+ \\ a_1x^2+2b_1xy+c_1y^2+2d_1x+2e_1y+f_1=0, \end{array}\right.$$

where A_1, B_1, ... a_1, b_1, ... are polynomials in A, B, ... a, b, ... and the eight parameters of the collineation. It is evidently possible to choose in an infinite number of ways the eight parameters in such a manner that in (2) the coefficients B_1, C_1, D_1, b_1, c_1 vanish, which amounts to five equations with eight unknown quantities. It is therefore possible to find a collineation transforming (1) into an equation of the form

$$y^2=\alpha x^3+\beta x^2+\gamma x+\delta,$$

or, resolving the right side into its linear factors,

$$(3) \qquad y^2 = \alpha(x-e_1)(x-e_2)(x-e_3),$$

in which e_1 has a different meaning from the e_1 used above.

The general equation of the cubic can therefore always be reduced to an equation of the form (3), so that the discussion of the cubic with respect to its type may be limited to equation (3). This equation represents a curve which is symmetrical with respect to the x-axis, and its shape depends essentially upon the values of e_1, e_2, e_3. Assume $e_1 \leqq e_2 \leqq e_3$. The following cases may be distinguished:

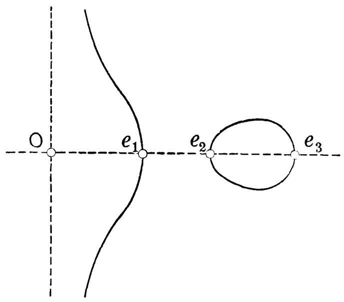


FIG. 80.

I. e_1, e_2, e_3 *are real and different from each other.*

On the x-axis the curve has the real points with the abscissas e_1, e_2, e_3, Fig. 80. In order that y^2 be positive, it is necessary that either $e_1 \leqq x \leqq e_2$ or $x \geqq e_3$. From this it follows easily that the cubic consists in this case of an oval and a serpentine. This is the case discussed in connection with the real quadruple.

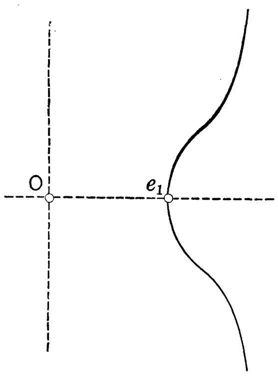


FIG. 81.

II. e_1 *is real,* e_2 *and* e_3 *are conjugate imaginary.*

In this case we can write (3) in the form

$$y^2 = \alpha(x-e_1)[(x-p)^2+g^2],$$

from which follows that y^2 is real only when $x \geqq e_1$; the curve consists of only one branch, Fig. 81. This case is equivalent with a fundamental quadruple with two real and two conjugate imaginary points, as we shall see later on.

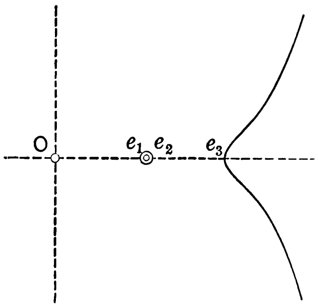


FIG. 82.

III. *$e_1 = e_2$ and e_3 all real.*

Equation (3) assumes the form

$$y^2 = \alpha(x - e_1)^2 (x - e^3).$$

To get real values for y, $x \geqq e_3$. The point $x = e_1$, $y = 0$ satisfies the equation also; but it is an isolated point, Fig. 82.

Correspondingly, in the quadruple two points are real and two coincide.

IV. *e_1 and $e_2 = e_3$ are real.*

The equation becomes

$$y^2 = \alpha(x - e_1)(x - e_2)^2.$$

y is real for $x \geqq e_1$. Hence $x = e_2$ is a double-point of the cubic, Fig. 83.

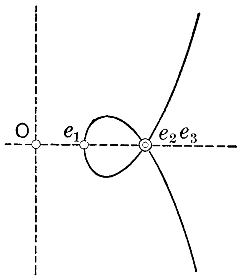


FIG. 83.

This case corresponds to a fundamental quadruple with two coincident and two conjugate imaginary vertices.

V. *$e_1 = e_2 = e_3$ and all real.*

Equation (3) can be written

$$y^2 = (x - e_1)^3.$$

We must take $x \geqq e_1$. The curve has a cusp at $x = e_1$ with the x-axis as a tangent, Fig. 84. The four points of the quadruple are real and three of them are coincident along the tangent of the cusp.

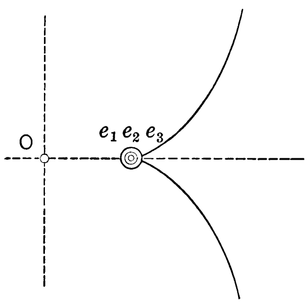


FIG. 84.

These are the five types of curves of the third order into which all cubics may be projected. NEWTON [1] called these five types, found by him, respectively,

parabola campaniformis cum ovali,

parabola pura,

parabola puncta,

[1] *Enumeratio linearum tertii ordinis* (Londini, 1706).

parabola nodata,

parabola cuspidata,

and, according to their behavior at infinity, subdivided them into seventy-two different kinds. By later investigations six more were added to the seventy-two.

As in the first case, this classification may be made by choosing in the perspective collineations the counter-axes r properly.

§ 49. Curves of the Third Order Generated by Involutoric Pencils.

1. Every straight line cuts a pencil of conics in an involution of points. Instead of any two conics $u=0$, $u_1=0$, we may take two degenerate conics with the same vertex:

$$u \equiv pp_1=0;$$
$$u_1\equiv(p+\lambda p_1)(p+\mu p_1)=0,$$

where p and p_1 represent two distinct straight lines. The pencil of conics then becomes

$$u+\nu u_1\equiv pp_1+\nu\{(p+\lambda p_1)(p+\mu p_1)\}=0,$$

where ν is a variable parameter. We may write this also in the form

$$\nu p^2+(1+\nu\mu+\nu\lambda)pp_1+\nu\lambda\mu p_1{}^2=0.$$

Solving for p,

$$p=\frac{-(1+\nu\lambda+\nu\mu)\pm\sqrt{(1+\nu\lambda+\nu\mu)^2-4\nu^2\lambda\mu}}{2\nu}\cdot p_1,$$

and designating by ξ and η the expressions multiplying p_1 in the last formula, the equation of the involutoric pencil of rays may be written

(1) $$(p-\xi p_1)(p-\eta p_1)=0,$$

where $\xi\eta=\lambda\mu=$ constant.

For every set of values of ξ and η satisfying this condition, (1) represents two rays of a pair of the involution

$$p-\xi p_1=0,$$
$$p-\eta p_1=0.$$

The product of two projective pencils of this kind, having the same ν,

(2) $$pp_1+\nu\{(p+\lambda p_1)(p+\mu p_1)\}=0,$$

(3) $$qq_1+\nu\{(q+\lambda' q_1)(q+\mu' q_1)\}=0,$$

is evidently the curve of the fourth order:

(4) $$pp_1\{(q+\lambda' q_1)(q+\mu' q_1)\}-qq_1\{(p+\lambda p_1)(p+\mu p_1)\}=0,$$

with the double-points $p=p_1=0$ and $q=q_1=0$.

In (1), $p=0$ and $p_1=0$ are evidently the equations of a pair of the involution.

In (3) the corresponding pair is given by $q=0$, $q_1=0$. Letting the corresponding rays p_1 and q_1 coincide; i.e., $p_1=q_1=0$, the curve (4) degenerates immediately into the ray $p_1=0$ and the cubic

(5) $$p\{(q+\lambda' p_1)\ (q+\mu' p_1)\}-q\{(p+\lambda p_1)\ (p+\mu p_1)\}=0.$$

To distinguish the pencils (2) and (3) from ordinary linear involutoric pencils, we shall call them quadratic. The result may be stated in the theorem:

The product of two projective quadratic involutions of rays is a curve of the fourth order. If the two involutions have two corresponding rays in common, then their product is a curve of the third order and that common ray.

The cubic can also be produced by two projective pencils of which one is linear and one quadratic:

(6) $$\begin{cases} p+\nu p_1, \\ qq_1+\nu\{(q+\lambda q_1)\ (q+\mu q_1)\}, \end{cases}$$

whose product is

$$(7) \qquad p\{(q+\lambda q_1)\ (q+\mu q_1)\}-p_1 q q_1=0.$$

As (5) and (7) contain respectively twelve and ten arbitrary parameters it is clear that every cubic may be represented by one of these forms.

2. Considering (5), each two pairs of corresponding rays of the two quadratic involutions (2) and (3), in which $q_1=p_1$,

$$(8) \qquad \begin{cases} p-\xi p_1=0, \\ p-\eta p_1=0, \end{cases}$$

$$(9) \qquad \begin{cases} q-\xi' p_1=0, \\ q-\eta' p_1=0, \end{cases}$$

intersect each other in four points of the cubic. The vertices of the two pencils are also on the cubic. Two pairs of the quadratic involution in one pencil and the two corresponding pairs in the other pencil are therefore sufficient to determine the projectivity and consequently also the cubic, since they determine ten points on the curve.

Conversely, if on a cubic two vertices B and B_1 are known, and if it occurs twice that two rays through B cut certain two rays through B_1 in four points of the cubic, then these pairs determine two projective quadratic involutions of rays by which the entire cubic is produced.

To prove this assume a ray, a, through B_1 passing very close to B. If the foregoing statement would not be true, the product of the involutions determined by the four pairs of rays through B and B_1 would be a curve of the fourth order, according to (4). On the ray a there would be two points cut out by the corresponding rays through B, which in general would be distinct. As the ray a in the limit approaches the ray through B_1 passing through B, these two points on a become coincident; i.e., B is a double-point of the curve (4). Similarly B_1 is also a double-point. A double-point on another curve

counts for two points of intersection, so that the supposed curve of the fourth order has twelve points in common with the given cubic. Construct the net of quartics (curves of the fourth order) through these twelve points. Any two points different from these twelve points, with these, determine fourteen points; i.e., such a quartic and only one, which therefore consists of the given cubic and the straight line through the assumed two points.

From this it follows that there is only one quartic through the twelve points having B and B_1 as double-points, and this consists of the given cubic and the line through B and B_1, which proves the proposition.

3. Consider three quadratic involutions of pencils of rays projective to each other:

$$pp_1+\nu\{(p+\lambda p_1)(p+\mu p_1)\}=0, \tag{10}$$

$$qq_1+\nu\{(q+\lambda' q_1)(q+\mu' q_1)\}=0, \tag{11}$$

$$rr_1+\nu\{(r+\lambda'' r_1)(r+\mu'' r_1)\}=0, \tag{12}$$

and suppose that the product of (10) and (11) is identical with the product of (10) and (12); i.e., that the two equations

$$pp_1\{(q+\lambda' q_1)(q+\mu' q_1)\}-\{(p+\lambda p_1)(p+\mu p_1)\}qq_1=0, \tag{13}$$

$$pp_1\{(r+\lambda'' r_1)(r+\mu'' r_1)\}-\{(p+\lambda p_1)(p+\mu p_1)\}rr_1=0, \tag{14}$$

must be simultaneously satisfied for all sets of values of x and y. This can only be true if

$$qq_1\{(r+\lambda'' r_1)(r+\mu'' r_1)\}-rr_1\{(q+\lambda' q_1)(q+\mu' q_1)\}=0 \tag{15}$$

simultaneously with (13) and (14).

From this the theorem follows:

If a quadratic involution of rays produces with two projective quadratic involutions one and the same quartic or cubic, then the product of the last two involutions is the same quartic or cubic.

4. Considering again the construction of a cubic by the Steinerian transformation, Fig. 85, and taking B at an infinite distance

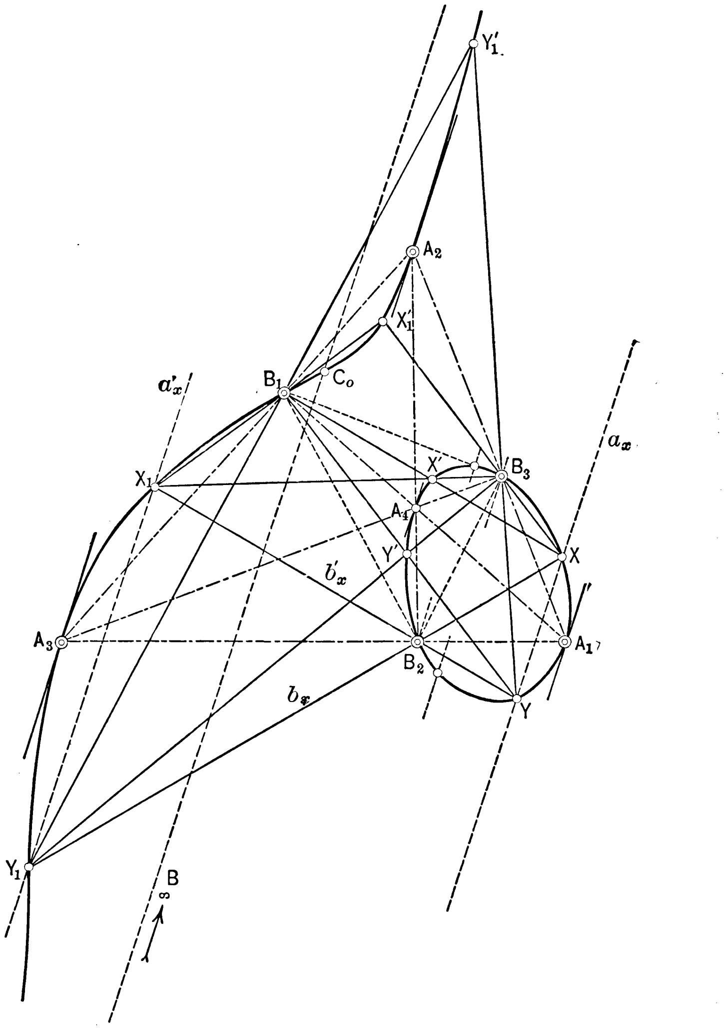


FIG. 85.

in the indicated direction, then to a point X on the cubic corresponds a point Y on a ray through X parallel to this direction. Joining

X and Y with B_2 and producing XB_2 and YB_2 to their intersections Y_1 and X_1 with the cubic, then X_1 corresponds to Y_1 in the Steinerian transformation and hence X_1Y_1 is parallel to XY. It is now clear that the pairs of rays XY_1, X_1Y through B_2 and XY, X_1Y_1 through B, furthermore the pairs A_1A_3 (counted twice) through B_2 and BA_1, BA_3, determine two projective quadratic involutions around B_2 and B whose product is a cubic, since they have the ray B_2B in common. This cubic having ten points in common with the cubic of the Steinerian transformation (A_1, A_3; each counted twice, since BA_1, BA_3 are tangents at A_1 and A_3; B, B_2, X, Y, X_1, Y_1) is identical with it. If we connect X and Y with B_3 and produce XB_3 and YB_3 to their intersections X_1' and Y_1' with the cubic, then X_1' and Y_1' correspond to each other in the Steinerian transformation; i.e., $X_1'Y_1' \parallel XY$. Consequently the cubic may also be considered as the product of the involution around B and an involution around B_3. In the same manner it is also the product of the involutions around B and B_1; hence, according to the foregoing theorem, the cubic is also the product of the two involutions around B_3 and B_1. Hence the points where X_1B_1 and Y_1B_1 produced meet the cubic are the same as X_1', Y_1'. In a similar manner it can be proved that XB_1 and X_1B_3, YB_1 and Y_1B_3 intersect each other in the points X', Y' of the cubic, so that $X'Y' \parallel XY$. X has been assumed as any point of the cubic, and X_1 in such a manner that the corresponding point Y of X lies in a straight line with X_1 and B_2. Consider now the pairs of rays XX_1', XB_1 and X_1X', X_1B_3; and XY, XY_1 and X_1Y_1, X_1Y; they determine two projective quadratic involutions about X and X_1 whose product is a cubic which is identical with the original cubic, since it has ten points in common with it. Taking any point G on the cubic [1] and letting XG and X_1G cut the cubic in J and K, then XJ and XK produced cut the cubic in one and the same point H; XG, XH and X_1G, X_1H form two corresponding pairs of the involutions around X and X_1. If G approaches the point of intersection of XX_1 with the cubic, then J

[1] For the sake of simplicity in the figure the following part of the construction is not shown.

and K approach X_1 and X; hence, in the limit, the tangents to the cubic at X and X_1 intersect each other in a point of the cubic. In a similar manner it can be proved that the tangents at Y and Y_1, X' and X_1', Y' and Y_1' intersect each other in points of the cubic. Again, G and H, and J and K may be assumed as vertices of projective quadratic involutions producing the cubic. Hence also the tangents at G and H, and J and K intersect each other in points of the cubic. We can therefore state the following theorem:

Designating two points on a cubic whose tangents at those points intersect each other in a point of the cubic as a Steinerian couple, or simply as a couple, then the cubic can be produced by two projective quadratic involutions around these points.

The lines joining any point of the cubic to the points of a couple cut the cubic again in a couple, and all couples of the cubic are produced when this point describes the whole cubic.

Each two corresponding pairs of the involutions around the two points of a couple intersect each other in two new couples. Such two involutions produce all couples of the cubic.

A quadruple on a cubic is defined as a group of four points any two of which form a couple; i.e., the tangents at the four points concur in a point of the cubic. From this definition we infer easily:

The lines joining four points of a quadruple cut the cubic in another quadruple. The sixteen lines joining the points of two quadruples intersect each other, four by four, in four points of a new quadruple.

These results form a part of the theory of problems of closure on the cubic as it has been developed by Steiner, Clebsch, and others.[1] They are sufficient for the applications in the following sections.

---

[1] For further details references are made to

CLEBSCH: *Crelle's Journal*, Vol. LXIII, pp. 94–121.

STEINER: *Crelle's Journal*, Vol. XXXII, pp. 371–373.

DISTELI: *Die Steiner'schen Schliessungsprobleme nach darstellend-geometrischer Methode.* Leipzig, 1888.

EMCH: *Applications of Elliptic Functions to Problems of Closure*, University of Colorado Studies, Vol. I, pp. 81–133.

Ex. 1. Verify the theorems of this section constructively, when B is finite or infinite.

Ex. 2. What relation must exist between a quadratic and a projective linear involution of rays in order that the cubic produced be one with a cusp.

Ex. 3. Prove directly that a cubic can be produced by two quadratic involutions around the points of a couple by determining two corresponding pairs of the involutions.

§ 50. Various Methods of Generating a Circular Cubic.

1. In § 48 (8), we found for the equation of the bicircular cubic referred to the equilateral triangle $A_1A_2A_3$ with A_4 as point of concurrence of altitudes, after some rearrangement,

$$(1) \qquad 4(\kappa x-y)(x^2+y^2)-2\kappa x^2-4xy+y+2y^2-2\kappa x+2y=0.$$

The slopes of the asymptotes at the circular points are evidently $+i$ and $-i$, so that the equations of these asymptotes are of the form

$$(2) \qquad \begin{cases} y=ix+c_1, \\ y=-ix+c_2, \end{cases}$$

where c_1 and c_2 are constants to be determined. If equations (2) represent asymptotes, then their common solutions with (1) must give infinite values for x and y. Substituting the values of y from (2) in (1), we get respectively

$$(3) \qquad \begin{cases} (8c_1+8c.\kappa i-4i-4\kappa)x^2+Bx+C \;=0, \\ (8c_2-8c_2\kappa i+4i-4\kappa)x^2+B'x+C'=0, \end{cases}$$

where B, C; B', C' are polynomials in c_1, κ; c_2, κ, different from those of the x^2's.

In both cases the values of x will be infinite, if we have respectively

$$8c_1+8c_1\kappa i-4i-4\kappa=0, \quad \text{or} \quad c_1=\frac{1}{2}\,\frac{\kappa+i}{1+\kappa i},$$

and

$$8c_2 - 8c_2\kappa i + 4i - 4\kappa = 0, \quad \text{or} \quad c_2 = \frac{1}{2}\frac{\kappa - i}{1 - \kappa i},$$

so that the equations of the asymptotes are

$$(4) \qquad \begin{cases} y = ix + \dfrac{1}{2}\dfrac{\kappa + i}{1 + \kappa i}, \\ y = -ix + \dfrac{1}{2}\dfrac{\kappa - i}{1 - \kappa i}. \end{cases}$$

Solving these two equations simultaneously, we get for the coordinates of the point of intersection of the two asymptotes (4)

$$(5) \qquad x = \frac{1}{2}\frac{\kappa^2 - 1}{\kappa^2 + 1}, \quad y = \frac{\kappa}{\kappa^2 + 1}.$$

The sum of the squares of these expressions is $x^2 + y^2 = \frac{1}{4}$; i.e., the point of intersection is on the circle corresponding to the infinite line in the Steinerian transformation. The real asymptote of the cubic has the slope κ, so that its equation is of the form $y = \kappa x + c_3$. By a similar method as in the case of c_1 and c_2 above, we find

$$c_3 = \frac{3\kappa - \kappa^3}{1 + 2\kappa^2},$$

and as the equation of the real asymptote

$$(6) \qquad y = \kappa x + \frac{3\kappa - \kappa^3}{1 + 2\kappa^2}.$$

Solving (1) and (6) simultaneously, we find for the point of intersection of this asymptote with the cubic

$$(7) \qquad x = \frac{1}{2}\frac{1 - \kappa^2}{1 + \kappa^2}, \quad y = -\frac{\kappa}{1 + \kappa^2}.$$

Comparing (7) with (5), it is seen that the two points are diametral.

A similar result is obtained by taking any orthogonal quadruple $A_1A_2A_3A_4$ and the circular cubic associated with it. In this case the equation of the cubic assumes the form

$$(8) \qquad (\alpha x+\beta y)(x^2+y^2)+ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

Repeating on this equation the same process as above on equation (1), the theorem may be stated thus:

Considering a bicircular cubic in a Steinerian transformation, the asymptotes at the circular points intersect each other in a point D of the circle which corresponds to the line at infinity in the Steinerian transformation. The real asymptote cuts the same circle in a point C which with D determines a diameter of the circle. The points D and C are called center and principal points of the cubic.

2. In equation (8) the infinitely distant real point of the cubic is the infinite point of the line $\alpha x+\beta y=0$, as can easily be verified. Taking the x-axis parallel to this line, (8) becomes

$$(9) \qquad y(x^2+y^2)+ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

In a similar manner as in (5), the coordinates of the center of the cubic are found to be

$$(10) \qquad x=b, \quad y=\frac{a-c}{2}.$$

Taking this point as the origin of a new coordinate system with axes parallel to those in (9), (9) assumes the form

$$(11) \qquad (y+a)(x^2+y^2)+2dx+2ey+f=0.$$

Here the equation of the real asymptote is $y=-a$, so that the coordinates of the principal point of the cubic become

$$x=\frac{2ae-f}{2d}, \quad y=-a.$$

Equation (11) may be considered as the result of the elimination of λ^2 from the two equations

$$x^2+y^2-\lambda^2=0,$$

$$2dx+2ey+f+\lambda^2(y+a)=0,$$

which represent two projective pencils of concentric circles and rays. Hence the theorem of CZUBER:[1]

Every circular cubic may be generated by two projective pencils of concentric circles and rays. The common center of all circles of the pencil is the center of the cubic, and the vertex of the pencil of rays is the principal point of the cubic.

3. In § 49 it has been shown that the points on a cubic may be arranged according to couples, so that the rays from any point on the cubic to the points of these couples form an involution.

Suppose now that the direction of the real asymptote of a circular cubic is perpendicular to one of the sides, say B_2B_3 of the diagonal triangle $B_1B_2B_3$, Fig. 86; then the center of the cubic will coincide with the point B_1. In other words, the circular points form a couple, so that the involutions of rays from any point of the cubic contain the directions of the circular points as a pair.

Hence, according to a theorem in § 5, p. 21, since A_1A_3, A_2A_4 are two couples and P any point on the cubic, the angles A_1PA_3 and A_2PA_4 are equal. Hence the theorem:

The circular cubic which contains the circular points as a couple (conjugate pair) is the locus of all points from which two fixed lines A_1A_3, A_2A_4 appear under the same angle.

Inscribing a conic to the quadrilateral $A_1A_2A_3A_4$, then the pieces A_1A_3 and A_2A_4 of the tangents contained between the two other tangents A_1A_2, A_3A_4 of the conic, are subtended by equal angles at the focus, § 35, p. 118. Hence the theorem:

[1] *Die Kurven dritter und vierter Ordnung, welche durch die unendlich fernen Kreispunkte gehen.* (Zeitschr. f. Math., XXXII, 1887.)

The locus of the foci of all conics inscribed to a quadrilateral is a circular cubic having the circular points as a conjugate pair.

The same cubic may also be produced by two projective pencils of circles over A_1A_3 and A_2A_4, in which two corresponding circles subtend equal peripheral angles over the chords A_1A_3 and A_2A_4.

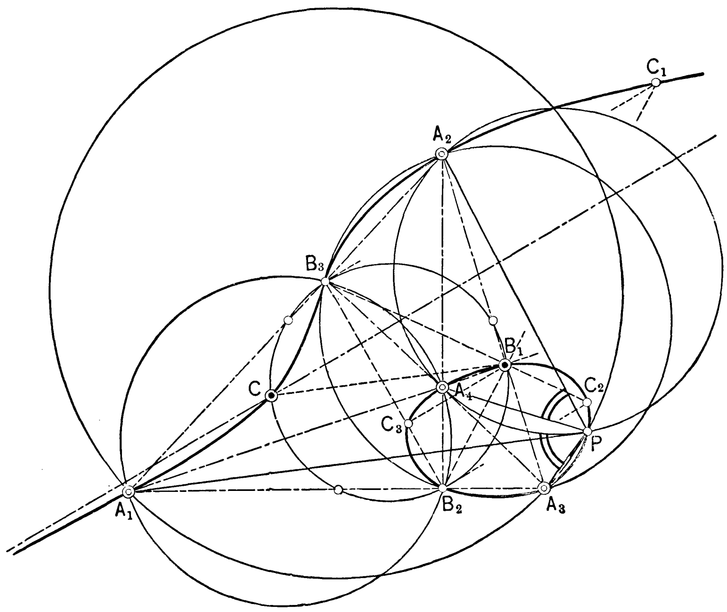


FIG. 86.

But if two projective pencils of circles $G+\lambda G_1=0$ and $G'+\lambda G_1'=0$ produce a cubic, say $GG_1'-G_1G'=0$, so that this equation reduces to $Gg_1'-G_1g'=0$, where g_1' and g' are linear factors, then the cubic may also be produced by two projective pencils of circles and rays.

4. In the same circular cubic consider the pencil of circles through B_2B_3, Fig. 86. The ray B_1C passes through the center of the circle through $B_1B_2B_3$. A_2A_3 passes through the center of the circle through A_2A_3 (diameter of said circle) and B_2B_3. $B_1A_4A_1$ passes through the center of the circle through A_4A_1 (diameter) and B_2B_3. The three circles through B_2B_3 and the

three corresponding rays through B_1 determine nine points of the cubic, since B_1 as a point of tangency on B_1C counts twice. The two pencils therefore determine two projective pencils of circles and rays whose product is the given cubic. Hence the theorem:

The circular cubic having the circular points as a conjugate pair is also the product of a pencil of circles and a projective pencil of rays which pass through the centers of the corresponding circles.

Ex. 1. With the Steinerian transformation for base, prove that the general equation of a circular cubic has the form

$$(\alpha x+\beta y)(x^2+y^2)+ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

Ex. 2. Given the pencil of circles

$$x^2+y^2-p^2-2\lambda x=0 \quad (p=\text{constant})$$

and a pencil of rays passing through the centers of these circles. To find the equation of the product of the two pencils and discuss the result for different positions of the vertex of the pencil.

Ex. 3. Establish the equations of two projective pencils of circles in which corresponding circles subtend equal peripheral angles over the fixed chords.

Ex. 4. Prove that in a circular cubic the oval and the serpentine appear under the same angle from any point of the curve.

Ex. 5. The extremities of two diameters A_1A_2 and A_3A_4 form a square. What is the locus of the points from which both diameters appear under the same angle?

§ 51. The Five Types of Cubics in the Steinerian Transformation.[1]

I. *Cubic with Oval and Serpentine.*

This cubic is obtained when all four points of the fundamental quadruple are either real or imaginary. As the case

---

[1] This section has been published in The University of Colorado Studies, Vol. I., No. 4, Feb. 1904.

with four real points has so far always been used to illustrate the general properties, we shall now assume an entirely imaginary quadruple determined by a coaxial system of circles with the limiting points P and Q, Fig. 87. On every ray g through an

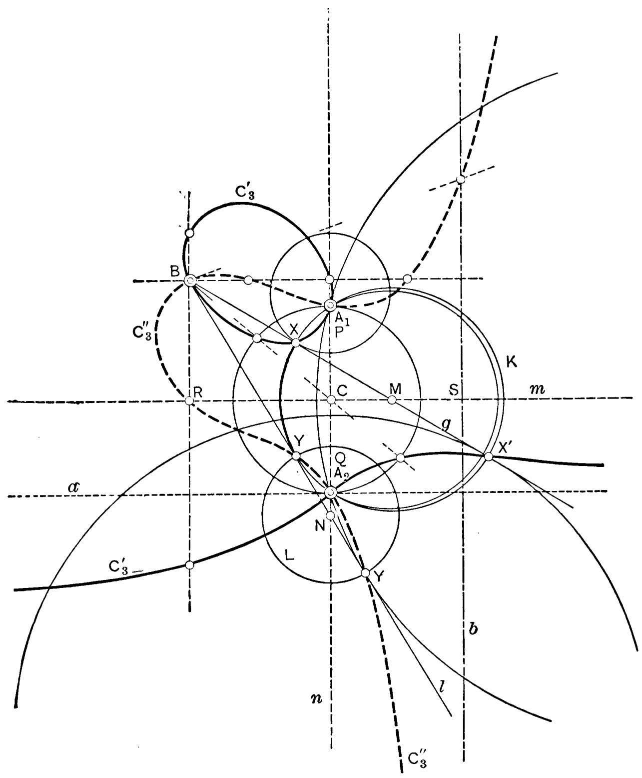


FIG. 87.

arbitrary fixed point B the circles of this system cut out an involution of points whose double-points X and X' are two points of the cubic associated with the point B in the Steinerian transformation belonging to the given imaginary quadruple. To construct X and X', let g cut the line m, which is the line joining the finite imaginary points of the quadruple, at M. With M as a center pass a circle K through P and Q which will cut g in the required

points. In reality, according to the well-known construction just explained X and X' are the points of tangency of g with two circles of the given system. It will be noticed from the figure that the two points of the cubic on a ray through B are always equally distant from m. Hence, taking a ray through B parallel to m, the point at infinity corresponding to Q will be in a line through P parallel to m. In other words, the line through P parallel to m is the asymptote of the cubic. Considering the pencil of circles through P and Q, the same circular cubic is also produced by this pencil and the pencil of diameters through B. Thus a statement in the foregoing section is corroborated.

II. *The Simple Cubic.*

This curve is produced by assuming two separate real and two imaginary points in the fundamental quadruple. In Fig. 87 let P and Q be the real points, and the circular points of the pencil of circles through P and Q the imaginary points. To find the points Y and Y' where a ray l through B cuts the cubic, let l cut n at N. With N as a center construct the circle L orthogonal to the pencil of circles through P and Q. The circle L cuts l in the required points Y and Y'. This cubic appears again plainly as the product of a pencil of circles and the pencil of diameters through B. Two points on a ray through B, like Y and Y', are always equally distant from n. To R corresponds the infinitely distant point of the cubic; consequently, the asymptote is parallel to n and its distance SC from n is equal to RC.

Ex. 1. Prove that the two cubics in Fig. 87 intersect each other orthogonally.

Ex. 2. Construct the tangents to the two cubics at B, P, Q.

III. *Cubic with an Isolated Point.*

The quadruple consists of two distinct points A_1, A_3 and two coincident points A_2, A_4. It is assumed that the direction of the line joining A_2 and A_4 in the limit; i.e., as they become coincident, cuts A_1A_3 at B_2. B_1 and B_3 will coincide with A_2 and A_4, Fig. 88. Joining B, which, as usual, has been assumed infinitely distant, to B_1, B_2, B_3, and constructing the fourth harmonic rays to the pairs of sides passing through these points, it is seen by

passing over to the limit that the fourth harmonic rays at B_1 and B_3 coincide. As before, they cut the fourth harmonic ray through B_2 at C, the point through which the asymptote passes.

The pencil of conics through the quadruple cuts every ray through B to the left of A_3 and to the right of A_1 in elliptic in-

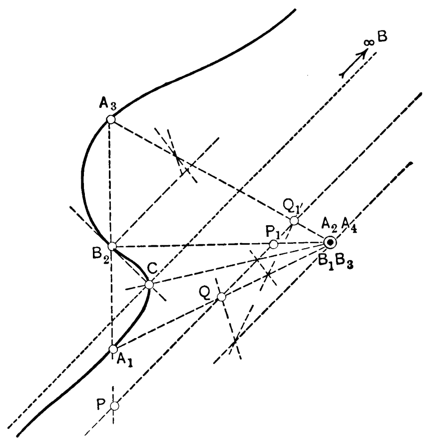


FIG. 88.

volutions, and only the rays between A_1 and A_3 contain hyperbolic involutions. The only branch of the cubic is therefore contained between two lines through A_1 and A_3 parallel to the direction of B. The ray through A_2A_4 carries a parabolic involution, and A_2A_4 represents an isolated point of the cubic.

IV. *Nodal Cubic.*

Assuming in the fundamental qnadruple A_1 and A_4 real and coincident and A_2, A_3 conjugate imaginary, a cubic with a double-point or node at A_1A_4 is obtained. In Fig. 89 a vertical line through A_1A_4 represents the limiting direction of the line joining the two points. As conics of the pencil through the fundamental quadruple take the pencil of circles tangent to each other at A_1A_4 and to the vertical line. A_2 and A_3 are then represented by the circular points at infinity. To construct the cubic associated with an arbitrary point B, draw rays through B. On each of these rays the pencil of circles cuts out an involution whose double-

points are points of the cubic. These points are, of course, also the points of tangency of circles of the pencil. Hence, to find the points where a ray g through B cuts the cubic, take the point M where g cuts m, the line joining A_1 with A_4, as a center of a circle K passing through A_1A_4. K cuts g in the required points X and X'. From this it is seen that the nodal cubic is also

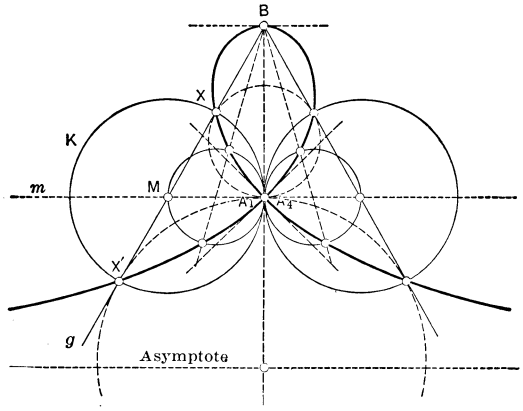


FIG. 89.

the product of a pencil of circles with coincident limiting points and a pencil of diameters. As X and X' are equally distant from m, the asymptote is parallel to m at a distance to the left of m equal to BA_1 ($BA_1 \perp m$ for the sake of symmetry.)

V. *Cuspidal Cubic.*

In this case three of the four real points of the fundamental quadruple coincide. Constructively this can be arranged best by assuming as the pencil of conics a pencil through a fixed point A_1 and with its conics all osculating each other at another fixed point which evidently may be considered as the representative of the three coincident points A_2, A_3, A_4.

To construct a pencil of osculating conics we may start with the construction of § 41, 9, Fig. 67. There it was shown that the picture of a circle in a perspective collineation whose

center lies on the axis of collineation and also on the given circle is a conic osculating the given circle at the center of collineation. Hence, considering in Fig. 90 the line s joining A_1 with the coincident remaining points as the common axis of an infinite number of perspective collineations in which only the

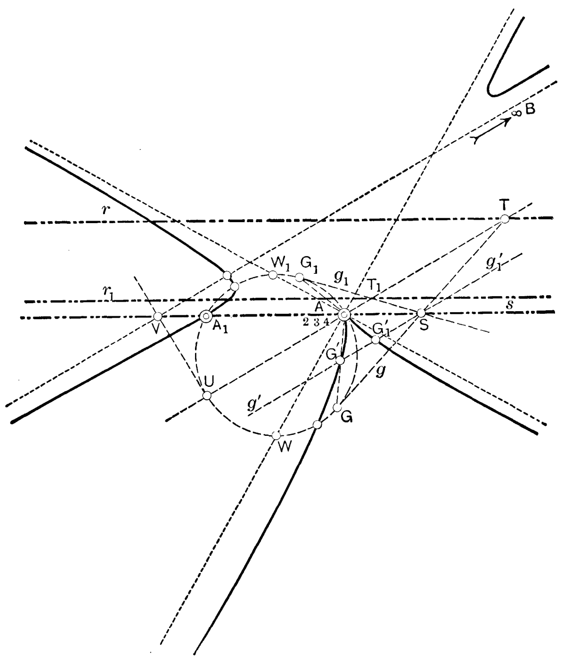


Fig. 90.

counter-axes vary, the pictures of a fixed circle K through $A_1A_2A_3A_4$ form clearly a pencil of osculating conics.

On every ray g' (or the identical g_1') through a fixed point B (assumed infinitely distant) this pencil cuts out an involution whose double-points are two points on the cuspidal cubic associated with B in the Steinerian transformation. These points are also the points of tangency of g' (g_1') with two conics of the pencil. For the actual construction of these points the following

simple method may be applied: Let g' intersect s at S. From S draw the two tangents g and g_1 to the circle K.[1] Through the center of collineation or the cusp draw a line l parallel to the direction of B. Let T and T_1 be the points of intersection of l with g and g_1, and through T and T_1 draw two lines r and r_1 parallel to s. Considering r and r_1 as counter-axes of two distinct collineations with the same axis s and the same center, then, according to the constructions of collineations, g' and g_1' will be the pictures of g and g_1 in these two collineations, and the rays joining A to G and G_1 cut g' (g_1') in two points G' and G_1' which evidently are the points of tangency with g' (g_1') of the two osculating conics corresponding to the fixed circle K in the two collineations (r, r_1). The line l cuts K at U; the tangent at U cuts s at V; and from the construction follows that the line through V parallel to l is an asymptote. In a similar manner the lines joining C to the points of tangency W and W_1 of the tangents to K parallel to s are the directions of the two other real asymptotes. By a suitable collineation this cuspidal cubic may be transformed into the symmetrical form of Newton's parabola cuspidata.

Ex. 1. Prove that if s is a diameter of K and the direction of B is perpendicular to s, then the above cubic degenerates into an equilateral hyperbola.

Ex. 2. Prove that if K is tangent to s, then the cubic degenerates into a parabola.

Ex. 3. A pencil of cubics is determined by two cubics or by eight arbitrary points of which no four are in the same straight line. But it is clear that the two cubics determining the pencil have nine points in common, hence all cubics of the pencil pass through these nine points. In other words: *All cubics passing through eight fixed points pass through a ninth fixed point.*

Ex. 4. Through four points A, B, C, D of a cubic draw the pencil of conics (K). Every conic K of this pencil cuts the cubic in two points P and Q. Prove that the secant PQ cuts the cubic in a fixed point.

---

[1] In Fig. 90 K is the only circle shown, and l is the line through A cutting this circle at U.

Ex. 5. Let two straight lines l and m cut a cubic in the points A, B, C and D, E, F. Construct the points of intersection G, H, I of AD, BE, CF with the cubic and prove that they are collinear.

Ex. 6. Construct the cubic in the Steinerian transformation when one of the points of the quadruple is infinitely distant.

CHAPTER V.

APPLICATIONS IN MECHANICS.

§ 52. A Problem in Graphic Statics.

1. Let 1, 2, 3, ... be a system of coplanar forces in a plane, Fig. 91*a*. With O and O' as poles construct two force-polygons, Fig. 91*b*, and in the previous figure the two corresponding funicular polygons. Considering in both figures the lines 0, 012, 0′12, 0′, 12, it is seen that corresponding lines are parallel and that they form in each case five sides of a complete quadrilateral. Hence, according to the last theorem of § 25, also the line joining the intersections of 0 and 0′, and of 012 and 0′12, in Fig. 91*a*, is parallel to $\overline{OO'}$ in Fig. 91*b*. In a similar way it can be proved that the line joining the intersections of 012 and 0′12, and of 0123 and 0′123, in Fig. 91*a*, is parallel to $\overline{OO'}$ in Fig. 91*b*, i.e., identical with the line joining (0 and 0′) with (012 and 0′12). This result may evidently be extended to any number of forces, so that we have the theorem:

Corresponding sides of two funicular polygons of a system of coplanar forces intersect each other in points of the same straight line.[1]

COROLLARY.—*If the forces are concurrent, they and the two funicular polygons determine a perspective collineation.*

2. The value of this theorem will appear from the solution of the following problem:

Two bars, AC and BC, connected by a pivot-point at C, are supported by pivots at A and B (Fig. 92). *Two forces, 1 and 2,*

[1] In Cremona's Graphic Statics this theorem follows from the fact that the two figures (*a* and *b*) form two reciprocal figures. See loc. cit., p. 127.

are applied to the bar AC, and in the same manner two forces, 3 and 4, to the bar BC. To find the reactions at the points A, B, C.

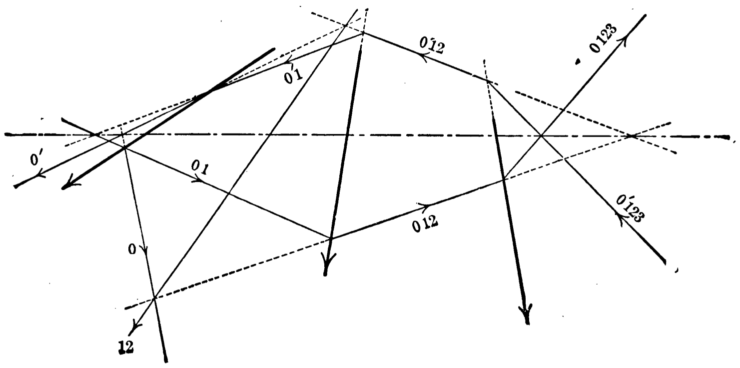


FIG. 91*a*.

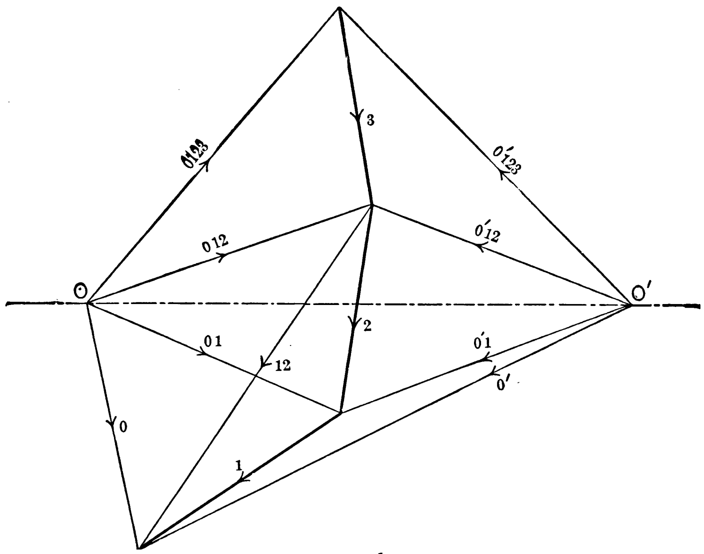


FIG. 91*b*.

First determine magnitude, direction, and position of the resultants (12) and (34) of the forces 1, 2 and 3, 4 by means of the polygon of forces (Fig. 92*b*) and the funicular polygon (Fig. 92*a*). Then construct the funicular polygon of the result-

ants (12) and (34) with O' as a pole and with its first side passing through A. Every funicular polygon constructed in this manner will be collinear with every other, and with the point of intersection M of (12) and (34) as the center of perspective

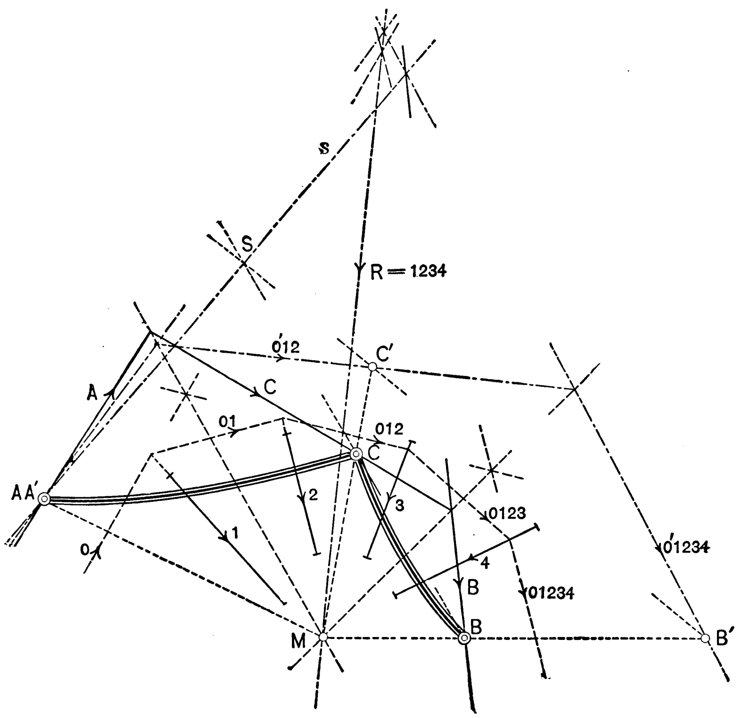


FIG. 92*a*.

collineation. Now it is clear that the polygon passing through A, C, and B, and formed by the reactions at these points, is also a funicular. It is therefore collinear with the first polygon (o′, o′12, o′1234). Projecting the points C and B from M upon the funicular sides (o′12) and (o′1234), respectively, the projected points C' and B' will correspond to C and B in a perspective collineation. Hence the lines BC and $B'C'$ will intersect each other

in a point S of the perspective axis s. By this point and the point A the axis is determined.

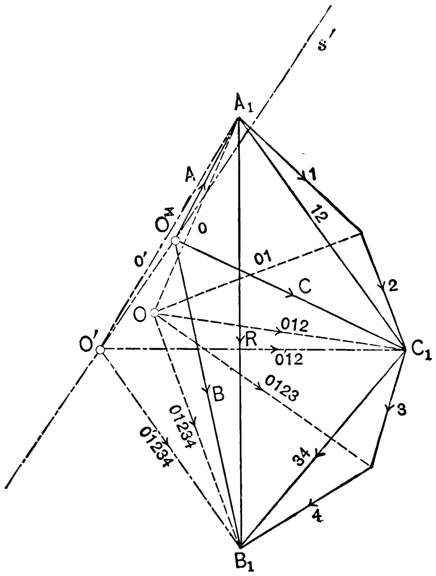


FIG. 92b.

The directions of the reactions at A, C, and B intersect the funicular lines o', o'12, o'1234 in points of the line s, and they may easily be drawn. To find the magnitudes of the reactions, draw lines parallel to their directions through the points A_1, C_1, B_1. These lines necessarily meet in a point O_1 of the straight line g'. Thus O_1A_1, O_1B_1, O_1C_1 are the magnitudes of the reactions at the points A, B, C.

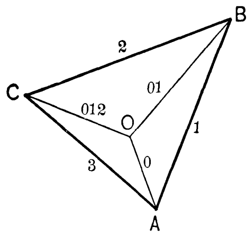
§ 53. Statical Proofs of Some Projective Theorems.

1. Constructing a funicular polygon of a system of coplanar forces 1, 2, 3, . . . , n, it is known that the resultant of the system passes through the point of intersection of the forces (o) and (o123 . . . n) of the funicular polygon, and is also the resultant

of these two extreme forces, with (0) reversed. Hence, when the system is in equilibrium, the forces (0) and (0123 . . . n) must coincide. Hence the theorem:

A funicular polygon of a system of coplanar forces in equilibrium is a closed figure.

Consider now three forces 1, 2, 3 in equilibrium, Fig. 93a, and draw any triangle 0, 01, 012 having its vertices on these forces. Draw also, in Fig. 93b, the force-polygon 123. Through the inter-



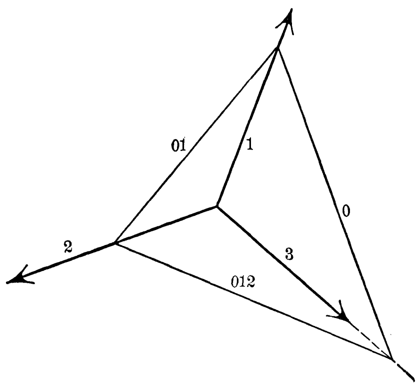


FIG. 93b. FIG. 93a.

section of 1 and 3 in (b) draw a line parallel to 0 in (a); through 1 and 2 one parallel to 01 in (a). These two lines intersect each other in a point O. Now connect O with the intersection of 2 and 3. Thus three forces OA, OB, OC are obtained, and if O is assumed as a pole and starting out with the original line 0 in (b), a funicular polygon is obtained which coincides with the original triangle 0, 01, 012. Hence the theorem:

Any triangle whose vertices lie on the lines of action of three forces in equilibrium may be considered as a funicular polygon of these forces.

Consequently, according to the theorem of § 51, if we take any two triangles with their vertices on the three lines of action,

the three points of intersection of corresponding sides are collinear. As any three concurrent lines may be chosen as lines of action of three forces in equilibrium, we thus have proved the well-known theorem concerning homologous triangles.

2. THEOREM. *The middle points of the diagonals of any complete quadrilateral are collinear.*[1]

MINCHIN[2] gives the following proof for this proposition: Let $ABCDEF$ be the complete quadrilateral. Take the following system of forces, supposed acting on a rigid body: two forces represented by DA and DC in magnitudes and senses, and two represented by BA and BC, Fig. 94

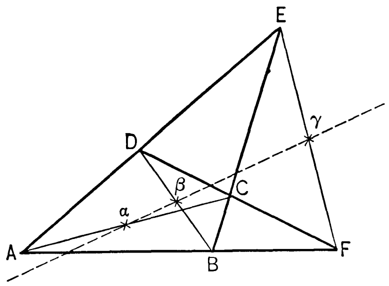


FIG. 94.

Now the resultant of the first pair passes through α, the middle point of AC; so does the resultant of the second pair; therefore the resultant of the four forces passes through α. Also the resultant of DA and BA passes through β, the middle point of BD; so does the resultant of DC and BC; hence the resultant of the four forces also passes through β. We shall now show that it passes through γ, the middle point of EF. For this purpose introduce a force ED and a force DE which do not alter the given system. Introduce also forces CE, EC; CF, FC; FB, BF. Hence the given system is equivalent to forces EA, AE; DF, DE; BE, BF; EC, FC; and it is obvious that the resultant of each of these pairs passes through γ; hence the resultant of the whole system passes through γ. Now as the resultant of the given system acts in a right line, and as α, β, γ have been independently shown to be points on this resultant, these points are collinear. Q.E.D.

3. Pascal's theorem, that the intersections of the opposite

[1] CHASLES, loc. cit., arts. 344, 345.
[2] *Treatise on Statics*, Vol. I, pp. 145, 146.

sides of a hexagon inscribed in a circle lie in a right line, is easily exhibited as a case of the funicular property in § 52. Following again Minchin, loc. cit., let the lines DA, EB, FC in Fig. 95 be lines of action of three forces, P, Q, R, such that if P is resolved at A into two components along AB, AF, or into two at D along DC, DE; if Q is resolved into two at B along BA, BC, or into two at E along ED, EF; and if R is resolved at C along CB, CD, or at F along FE, FA, the two components thus obtained along any side are equal and opposite. Obviously such conditions are consistent, on account of the equality of angles in the same segment of a circle. Now if P, Q, R are applied at A, B, C, by the nature of the case a polygon $FABCD$ of jointed bars pivoted at F and D would be kept in equilibrium; i.e., this is a funicular of the forces. Again, let P, Q, R be applied at D, E, F to a polygon $CDEFA$ of jointed bars pivoted at C and A. This polygon would be in equilibrium, and a funicular of the forces. The two funiculars of the same forces, however, have the property that the intersections, α, β, γ, of their corresponding sides (AB, DE), (BC, EF), (CD, FA) are collinear. Q.E.D.

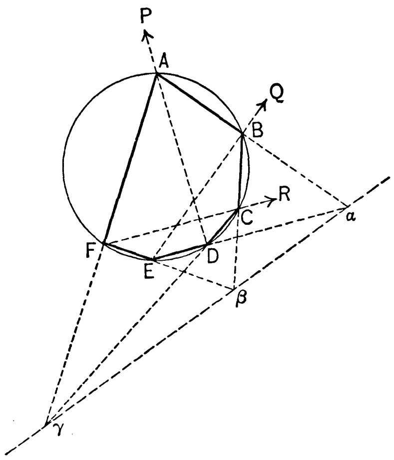


FIG. 95.

Ex. Prove that the medians of a triangle are concurrent.

GEOMETRY OF STRESSES IN A PLANE.

§ 54. General Remarks.

Forces acting on a body cause certain displacements or strains between its particles. These strains are said to be within the

elastic limit if after the disappearance of these forces the strains disappear also; i.e., if the body returns to its original condition. The forces which occur within the body as a result of the strains are called stresses. These are called *tensions*, *compressions*, or *shears*, according as their tendency is to pull the particles apart, to press them together, or to push them over one another. According to Hooke's law the stresses in a body are approximately proportional to the corresponding strains as long as they occur within the elastic limit. In many cases, plane surfaces may be passed through strained bodies orthogonally to which there are no strains and consequently no stresses. This is the case in beams under tension, compression, or bending, and covers a great number of engineering structures. In these cases the investigation of strains and stresses is limited to the plane. In what follows only stresses in a plane will be considered.

The forces producing the stresses in a body and these themselves are in equilibrium. The stresses in any portion of the solid are also in equilibrium. Considering an infinitesimal plane section in a strained solid, we make the assumption that the stresses acting on this element are uniformly distributed, so that their resultant passes through the center of gravity of this element. For many purposes it is convenient to consider the resultant stress per unit of the surface-element. This stress, the resultant divided by the element, is called the *specific stress* acting on that element.

§ 55. Involution of Conjugate Sections and Stresses.

1. Calling a plane surface through a strained body with the stresses acting in this plane (no stresses normal to the plane, as assumed above) a field of forces, we assume that under the influence of this field every portion of this plane is in equilibrium. Thus, if a very small triangle ABC (infinitesimal in all rigor) is cut out of the field, the resultant stresses acting upon its sides must be in equilibrium. According to the assumption of the uniform distribution of stresses over an infinitesimal section,

these resultants must pass through the middle points γ, α, β of the sides $AB=c$, $BC=a$, $CA=b$, and, being in equilibrium, are concurrent. Designate these resultants by A, B, C, as shown in Fig. 96. Each two of these forces, for instance A and B, may be resolved into components parallel to the sides AC and BC. Let A_1, A_2 and B_1, B_2 be these components. A_2 and B_2 act along the sides BC and AC, while $A_1 \parallel AC$ and $B_1 \parallel BC$. As A_1 and B_1 both pass through γ, their resultant will pass through γ.

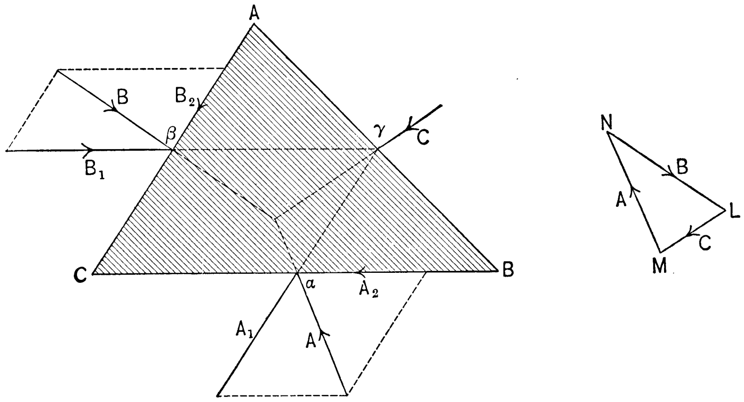


FIG. 96.

In consequence, the resultant of A_2 and B_2, which passes through C, must pass through γ, since C is the resultant of A_1, B_1 and A_2, B_2. Now $C\gamma$ is half the diagonal in the parallelogram having AB as the other diagonal. In order that the resultant of A_2 and B_2 lies in the diagonal $C\gamma$ the proportion

$$\frac{A_2}{a}=\frac{B_2}{b}$$

must hold. $\frac{A_2}{a}$ and $\frac{B_2}{b}$, however, are the specific stresses acting along the sections BC and AC. Hence the theorem:

If the specific stresses acting on two different sections at a point are each resolved into components parallel to these sections, then the components acting along these sections are equal.

If the sections AC and BC are perpendicular to each other, A_1 and B_1, A_2 and B_2 represent the normal and transversal components of the stresses. Hence the corollary:

The transversal stresses acting on two perpendicular sections are equal.

Taking at C any section CB and the stress A acting on it and drawing another section CA parallel to A, we have $A_1=A$, $B_1=B$, $A_2=B_2=0$. This gives the corollary:

If the force A acting on a section CB is known, then the force B, acting on a section CA parallel to A, is parallel to CB.

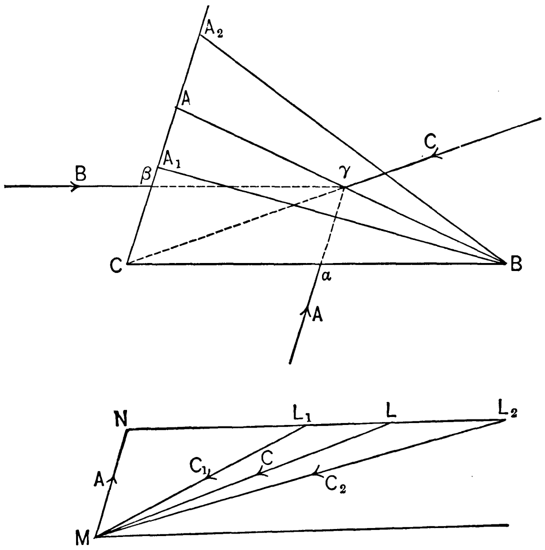


FIG. 97.

2. According to this corollary for every section through a point C there exists a certain stress acting on this section, such that if the direction of this stress is considered as a section, the stress belonging to this section acts along the original section. This, however, is a clear *expression for the involutoric character of the directions of sections and corresponding stresses.*

To prove this directly, keep in Fig. 97 the stress A and the

direction of the stress $B \parallel CB$ constant, and let the section BA turn about the fixed point B. The extremity A of BA traces on the section CA a point-range $AA_1A_2 \ldots$, so that the corresponding stresses $B, B_1, B_2 \ldots$ are proportional to the distances $CA, CA_1, CA_2, \ldots$. In the force-polygon the extremities of the B-stresses are marked by $LL_1L_2 \ldots$, and the corresponding C-stresses are $ML, ML_1, ML_2, \ldots$. Now the distances $NL, NL_1, NL_2, \ldots$ are proportional to the distances $CA, CA_1, CA_2, \ldots$; hence the projectivity of the pencils

$$(B \cdot AA_1A_2 \ldots) \barwedge (M \cdot LL_1L_2 \ldots).$$

Moving these pencils parallel to themselves so that M coincides with B, we have at B an involutoric pencil of sections and directions of corresponding stresses.

§ 56. Discussion of this Involution.

In Fig. 97 the sections CA and CB are both acted upon by compressions; in consequence the stress acting on the section is a compression. From the figure it appears clearly that corresponding rays of the involution in this figure move in the same direction. Hence, according to § 3, the involution is elliptic. The same is true if there are only tensions. In these cases there are no double-rays, i.e., *there are no sections where there are only shearing (transversal) stresses. In all sections the material is either entirely under the influence of compressions or under the influence of tensions.*

As every involution in a pencil admits of two rectangular rays, it follows that *through every point of a plane of stresses there are two sections on which only normal stresses are acting. In case of elliptic involutions these normal stresses are either both compressions or both tensions.*

If in Fig. 97 one section, say CA, is acted upon by a tension and the other, CB, by a compression, it is apparent that corresponding rays of the involution (§ 3) move in opposite directions.

The involution is hyperbolic and has two real double-rays (sections) in which only shearing stresses are acting.

Considering two corresponding rays BA and C, they are always separated by one of the double-rays, say d_1, Figs. 98 and 99. If a compression acts on BA, it will be so when BA approaches

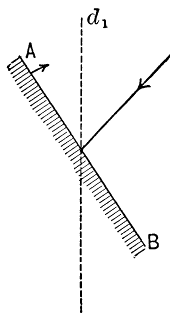


Fig. 98.

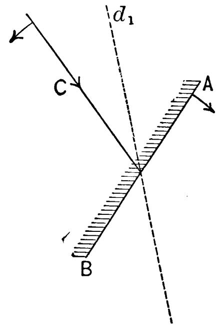


Fig. 99.

d_1. But as soon as these corresponding rays have crossed the double-ray d_1, the section AB is acted upon by tension. From this it follows that *the material included by one angle formed by the double-rays is subject to tension only, while the supplementary part is subject to compression only.*

As the angles formed by the double-rays are bisected by the rectangular pair, it follows that *the stress acting on one section where there is only a normal stress is a compressive force, while the stress acting on the perpendicular section is a tensile force.*

For a further discussion of these involutions and their extension to space we refer to Ritter's *Graphische Statik*, Vol. I, pp. 1–46, published by Meyer & Zeller, Zürich.

§ 57. The Stress Ellipse.[1] Metric Properties of the Involution of Stresses.

1. According to the previous section the specific stress acting on every section through a fixed point can be constructed as soon as the specific stresses acting on any two sections are known. In Fig. 100 assume these two sections, CA and CB, parallel to the x- and y-axis of a Cartesian system, and let the variable section AB include an angle α with the positive part of the x-axis. As in Fig. 96, resolve the stresses A and B into transversal and

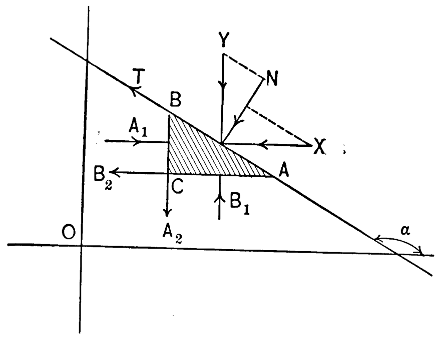


FIG. 100.

normal components A_2, B_2 and A_1, B_1. Designating the specific stresses determined by these components by t_a, t_b and n_a, n_b, we have $A_1 = a \cdot n_a$, $B_1 = b \cdot n_b$, $A_2 = a \cdot t_a$, $B_2 = b \cdot t_b$. Evidently $t_a = t_b = t$, say. The resultant C of A and B can also be resolved into transversal and normal components T and N with the corresponding specific stresses τ and ν, so that $T = c\tau$, $N = c\nu$. Let X and Y be the components of C parallel to the coordinate axes. Desig-

[1] Elegant graphic constructions for stress-ellipses may be found in RITTER'S *Graphische Statik*, loc. cit. For a thorough analytic discussion see M. LÉVY'S *Statique Graphique*, Vol. I, pp. 527–548 (Note IV).

nating the specific stresses by ξ and η, $X=c\xi$, $Y=c\eta$. From the figure we have

$$\begin{aligned}
&a=c \sin \alpha, \quad b=-c \cos \alpha, \\
&X=A_1+B_2=an_a+b\cdot t_b, \quad \text{or} \\
&X=c(n_a \sin \alpha-t_b \cos \alpha) \quad \text{and} \\
&\xi=n_a \sin \alpha-t_b \cos \alpha. \\
&Y=B_1+A_2=bn_b+at_a, \quad \text{or} \\
&Y=-c(n_b \cos \alpha-t_a \sin \alpha) \quad \text{and} \\
&\eta=-n_b \cos \alpha+t_a \sin \alpha.
\end{aligned}$$

Now

$$\begin{aligned}
&\nu=-\eta \cos \alpha+\xi \sin \alpha, \quad \text{hence} \\
&\nu=n_b \cos^2 \alpha-t_a \sin \alpha \cos \alpha+n_a \sin^2 \alpha-t_b \sin \alpha \cos \alpha, \quad \text{or} \\
&\nu=\tfrac{1}{2}(n_a+n_b)+\tfrac{1}{2} \cos 2\alpha\cdot(n_b-n_a)-\tfrac{1}{2} \sin 2\alpha\cdot(t_b+t_a).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\tau=\xi \cos \alpha+\eta \sin \alpha, \quad \text{or} \\
&\tau=n_a \sin \alpha \cos \alpha-t_b \cos^2 \alpha-n_b \sin \alpha \cos \alpha+t_a \sin^2 \alpha, \quad \text{or} \\
&\tau=\tfrac{1}{2}(n_a-n_b) \sin 2\alpha+\tfrac{1}{2}(t_a-t_b)-\tfrac{1}{2}(t_a+t_b) \cos 2\alpha.
\end{aligned}$$

As $t_a=t_b=t$, we have finally

(1) $$\nu=\tfrac{1}{2}(n_a+n_b)-\tfrac{1}{2}(n_a-n_b) \cos 2\alpha-t \sin 2\alpha,$$

(2) $$\tau=\tfrac{1}{2}(n_a-n_b) \sin 2\alpha-t \cos 2\alpha.$$

Designating the angle which the direction of C makes with the positive x-axis by β, we have

$$\tan \beta=\frac{\eta}{\xi}=\frac{-n_b \cos \alpha+t_a \sin \alpha}{n_a \sin \alpha-t_b \cos \alpha},$$

or

$$\tan \beta=\frac{t \tan \alpha-n_b}{n_a \tan \alpha-t}.$$

Solving for tan α, we get

$$\tan\alpha=\frac{t\tan\beta-n_b}{n_a\tan\beta-t}, \tag{3}$$

which clearly shows the involutoric character between the directions of a section and the stress acting on this section. This is in agreement with the geometric discussion of stresses in § 55.

For the double-elements of this involution we have

$$\tan\beta=\tan\alpha=m,$$

$$n_a m^2-2tm+n_b=0,$$

$$m=\frac{t\pm\sqrt{t^2-n_an_b}}{n_a}. \tag{4}$$

These values for m are real when n_a and n_b have different signs; in this case the involution is hyperbolic. If n_a and n_b both have the same sign, and $n_an_b>t^2$, then the involution is elliptic. For $n_an_b=t^2$ the involution is parabolic and $\tan\alpha=\frac{t}{n_a}=$ const.; i.e., the stresses all act in the same constant direction. This is the case in a rod under tension or compression exclusively.

2. Letting ν_1 and ν_2 be the normal specific stresses on two perpendicular sections determined by the angles α and $\alpha-\frac{\pi}{2}$, and $\pm\tau_\alpha$ the transversal specific stresses in these sections, from the formulas for ν and τ we get

$$\tau_\alpha{}^2-\nu_1\nu_2=t^2-n_an_b=\text{const.}$$

To get the rectangular pair of the involution, we form

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\dfrac{t\tan\alpha-n_b}{n_a\tan\alpha-t}}{1+\tan\alpha\dfrac{t\tan\alpha-n_b}{n_a\tan\alpha-t}}.$$

In this $\alpha-\beta=\frac{\pi}{2}$, if $1+\tan\alpha\cdot\frac{t\tan\alpha-n_b}{n_a\tan\alpha-t}=0$, or

$$t\tan^2\alpha+(n_a-n_b)\tan\alpha-t=0, \quad \text{or}$$

$$\tan\alpha=\frac{n_b-n_a\pm\sqrt{(n_b-n_a)^2+4t^2}}{2t},$$

an expression which is always real.

From this, and also from the expression for $\tau=0$, follows

$$(5) \qquad \tan 2\alpha=\frac{2t}{n_a-n_b}.$$

3. We shall now find the locus of the extremities of the specific stresses $\frac{C}{c}$ acting on all sections. Its coordinates are evidently ξ and η when referred to the point of application as an origin. From

$$(6) \qquad n_a\sin\alpha-t_b\cos\alpha=\xi,$$

$$(7) \qquad t_a\sin\alpha-n_b\cos\alpha=\eta,$$

the expressions for $\sin\alpha$ and $\cos\alpha$ result:

$$\sin\alpha=\frac{n_b\xi-t\eta}{n_an_b-t^2},$$

$$\cos\alpha=\frac{t\xi-n_a\eta}{n_an_b-t^2};$$

and since $\sin^2\alpha+\cos^2\alpha=1$, the required equation

$$(8) \qquad \xi^2(n_b^2+t^2)-2\xi\eta(n_a+n_b)t+\eta^2(n_a^2+t^2)-(n_an_b-t^2)^2=0$$

results. In this equation

$$(n_a^2+t^2)(n_b^2+t^2)-(n_a+n_b)^2t^2=(n_an_b-t^2)^2>0;$$

it represents, therefore, an ellipse, the so-called *stress-ellipse*.

From analytical geometry, § 31, the angles θ and $\theta+\frac{\pi}{2}$ of the axes of this ellipse are determined by

$$(9)\qquad \tan 2\theta=\frac{-2t(n_a+n_b)}{n_b{}^2+t^2-(n_a{}^2+t^2)}=\frac{2t}{n_a-n_b}.$$

Hence, according to (5), we have the theorem:

The axes of the stress-ellipse coincide with the rectangular pair of the involution of stresses around the center of the ellipse.

From

$$\nu=\tfrac{1}{2}(n_a+n_b)-\tfrac{1}{2}(n_a-n_b)\cos 2\alpha-t\sin 2\alpha$$

we find, by differentiation with respect to α, the condition for maximal and minimal normal stresses,

$$t\sin 2\alpha(n_b-n_a)-2t\cos 2\alpha=0,$$

or

$$(10)\qquad \tan 2\alpha=\frac{2t}{n_a-n_b}.$$

Hence, according to (5), the theorem:

The maximal and minimal normal stresses occur on the sections of the rectangular pair of the involution, or on the axes of the stress-ellipse. In these sections $\tau=0$.

In a similar manner, from

$$\tau=\tfrac{1}{2}(n_a-n_b)\sin 2\alpha-t\cos 2\alpha$$

we find for the maximal and minimal shearing stresses the condition

$$(11)\qquad \tan 2\alpha'=-\frac{n_a-n_b}{2t};$$

hence, from (10) and (11),

$$\tan 2\alpha\cdot\tan 2\alpha'=-1,\quad 2\alpha=2\alpha_1\pm\frac{\pi}{2},\quad \alpha'=\alpha\pm\frac{\pi}{4},\quad \text{or:}$$

The directions of the maximal and minimal shearing stresses bisect the angles formed by the maximal and minimal normal stresses and are equal (except as to sign).

4. The directions β_1 and β_2 of the stresses corresponding to two rectangular sections with the inclinations α and $\alpha+\frac{\pi}{2}$ are, according to (3), determined by

$$\tan\beta_1=\frac{t\tan\alpha-n_b}{n_a\tan\alpha-t}, \tag{12}$$

$$\tan\beta_2=\frac{t\cot\alpha+n_b}{n_a\cot\alpha+t}. \tag{13}$$

From (12), $\tan\alpha=\dfrac{t\tan\beta_1-n_b}{n_a\tan\beta_1-t}$, $\cot\alpha=\dfrac{n_a\tan\beta_1-t}{t\tan\beta_1-n_b}$.

Substituting this in (13) and reducing, we get

$$\tan\beta_2=\frac{t(n_a+n_b)\tan\beta_1-(n_b{}^2+t^2)}{(n_a{}^2+t^2)\tan\beta_1-t(n_a+n_b)}. \tag{14}$$

From this formula follows at once:

The directions of pairs of stresses corresponding to pairs of perpendicular sections form an involution.

This involution is identical with the involution of conjugate diameters of the stress-ellipse.

To prove the second part of this theorem, form the equation of the polar

$$\xi\xi_1(n_b{}^2+t^2)-(\xi_1\eta+\xi\eta_1)(n_a+n_b)t+\eta\eta_1(n_a{}^2+t^2)-(n_an_b-t^2)^2=0 \tag{15}$$

for any point (ξ_1, η_1), for which $\frac{\eta_1}{\xi_1}=\tan\beta_1$, with respect to the stress-ellipse.

For the point infinitely distant in the direction of β_1 there still is $\frac{\eta_1}{\xi_1}=\tan\beta_1$ and $\xi_1=\infty$, $\eta_1=\infty$. Hence

$$\xi(n_b{}^2+t^2)-(\eta+\xi\tan\beta_1)(n_a+n_b)+\eta\tan\beta_1(n_a{}^2+t^2)=0,$$

or

$$\frac{\eta}{\xi}=\frac{t(n_a+n_b)\tan\beta_1-(n_b{}^2+t^2)}{(n_a{}^2+t^2)\tan\beta_1-t(n_a+n_b)},$$

which is nothing else than $\tan\beta_2$ in (14), Q.E.D.

5. The normal stresses, ν_1, ν_2, on two perpendicular sections, determined by the angles α and $\alpha-\frac{\tau}{2}$, are

$$\nu_1=\tfrac{1}{2}(n_a+n_b)-\tfrac{1}{2}(n_a-n_b)\cos 2\alpha-t\sin 2\alpha,$$

$$\nu_2=\tfrac{1}{2}(n_a+n_b)+\tfrac{1}{2}(n_a-n_b)\cos 2\alpha+t\sin 2\alpha.$$

Adding, we get

$$\nu_1+\nu_2=n_a+n_b=\text{const.} \tag{16}$$

Hence the theorem:

The sum of the normal stresses acting on two perpendicular sections is constant and equal to the sum of the maximal and minimal normal stresses.

6. Between the angles β and α which the directions of a section and the corresponding stress make with the positive x-axis the involutoric relation

$$\tan\beta=\frac{t\tan\alpha-n_b}{n_a\tan\alpha-t} \tag{17}$$

exists. The central ray of the involution, for which $\beta=0$, is determined by $\tan\alpha=\frac{n_b}{t}$. Designate this value of α by γ, so that $\frac{n_b}{t}=\tan\gamma$. Take a line parallel to the ray for which $\beta=0$, at a distance p from it, and project the involution of stresses on this line. Then, from Fig. 101, $AC\cdot BC=\text{const.}$ To determine this constant, we have $BC=BD-CD$, $AC=CD+DA$; hence

$$AC\cdot BC=-\left(\frac{p}{\tan\beta}-\frac{p}{\tan\gamma}\right)\left(\frac{p}{\tan\gamma}-\frac{p}{\tan\alpha}\right),$$

or, after reducing,

$$(18) \qquad AC \cdot BC = -p^2 \frac{n_a n_b - t^2}{n_b^2} = k, \text{ say.}$$

In a similar manner, for the constant of the involution of conjugate diameters of the stress-ellipse we find

$$(19) \qquad A_1C_1 \cdot B_1C_1 = -p^2 \frac{(n_a n_b - t^2)^2}{(n_b^2 + t^2)^2} = k_1, \text{ say.}$$

Assuming that n_a and n_b are normal stresses on two perpendicular sections, then $t=0$. Without loss of generality we may

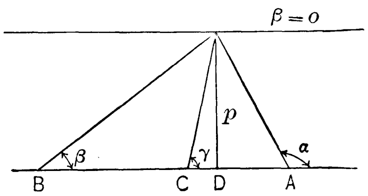


FIG. 101.

also assume $p=1$, so that (18) and (19) become

$$(20) \qquad k = -\frac{n_a}{n_b}, \quad k_1 = -\left(\frac{n_a}{n_b}\right)^2 = -k^2,$$

where n_a and n_b now designate the maximal and minimal normal stresses. Hence, when the stress-ellipse is known, it is not difficult to construct the involution of stresses.

§ 58. Examples.

1. A plane linear deformation is defined by the equations

$$(1) \qquad \begin{cases} x' = ax + by, \\ y' = a_1x + b_1y. \end{cases}$$

Referring the points (x, y) and (x', y') to an oblique system of coordinates (ξ, η) having the same origin and whose axes include the angles α and α_1 respectively, we have

$$(2) \qquad \begin{cases} x = \xi \cos\alpha + \eta \cos\alpha_1, \\ y = \xi \sin\alpha + \eta \sin\alpha_1. \end{cases}$$

Applying this to the points (x, y) and (x', y'), we get, according to (1),

$$\xi' \cos\alpha + \eta' \cos\alpha_1 = \xi(a\cos\alpha + b\sin\alpha) + \eta(a\cos\alpha_1 + b\sin\alpha_1),$$

$$\xi' \sin\alpha + \eta' \sin\alpha_1 = \xi(a_1\cos\alpha + b_1\sin\alpha) + \eta(a_1\cos\alpha_1 + b_1\sin\alpha_1),$$

or, solving for ξ' and η',

$$\xi' = \frac{1}{\sin(\alpha_1 - \alpha)}\{\xi(a\cos\alpha\sin\alpha_1 + b\sin\alpha\cdot\sin\alpha_1 - a_1\cos\alpha\cos\alpha_1 - b_1\sin\alpha\cos\alpha_1) + \eta[b\sin^2\alpha_1 - a_1\cos^2\alpha_1 + (a - b_1)\sin\alpha_1\cos\alpha_1]\},$$

$$\eta' = \frac{1}{\sin(\alpha - \alpha_1)}\{\xi[b\sin^2\alpha_1 - a_1\cos^2\alpha + (a - b_1)\sin\alpha\cos\alpha] + \eta(a\cos\alpha_1\sin\alpha + b\sin\alpha_1\sin\alpha - a_1\cos\alpha_1\cos\alpha - b_1\sin\alpha_1\cos\alpha)\}.$$

In these expressions we can choose the angles α and α_1 in such a manner that $\tan\alpha$ and $\tan\alpha_1$ are the roots of

$$b\cdot\tan^2\alpha + (a - b_1)\tan\alpha - a_1 = 0,$$

so that

$$\tan\alpha = \frac{b_1 - a + \sqrt{(b_1 - a)^2 + 4a_1 b}}{2b},$$

$$\tan\alpha_1 = \frac{b_1 - a - \sqrt{(b_1 - a)^2 + 4a_1 b}}{2b}.$$

Under these conditions the coefficients of η and ξ in the expressions for ξ' and η', respectively, vanish and the linear transformation in this system of oblique coordinates assumes the form

$$(3)\quad \begin{cases} \xi' = \dfrac{-2(ab_1 - 2a_1b) + (a^2 + b_1{}^2) + (a + b_1)\sqrt{(b_1 - a)^2 + 4a_1b}}{\sqrt{(b_1 - a)^2 + 4a_1b}} \cdot \xi \\ \eta' = \dfrac{2(ab_1 - 2a_1b) - (a^2 + b_1{}^2) + (a + b_1)\sqrt{(b_1 - a)^2 + 4a_1b}}{\sqrt{(b_1 - a)^2 + 4a_1b}} \cdot \eta. \end{cases}$$

From these formulas follows that the linear deformation (1) may be considered as two consecutive stretches along two oblique axes or directions. The angle ϕ formed by these axes is determined by

$$\tan \phi = \frac{\sqrt{(b_1 - a)^2 + 4a_1b}}{b - a_1}.$$

2. Evidently the rectangular transformation

$$(4)\quad \begin{cases} x' = ax, \\ y' = b_1y \end{cases}$$

is a special case of (3).

The elongations along the x- and y-axes are

$$\frac{x' - x}{x} = a - 1 \quad \text{and} \quad \frac{y' - y}{y} = b_1 - 1.$$

We can also write (4) in the form

$$(5)\quad \begin{cases} x' = x + (a - 1)x, \\ y' = y + (b_1 - 1)y. \end{cases}$$

Consider (5) as the analytical expression of a strain in a very thin plate which is assumed to have the property of a perfect

solid. Then $a-1$ and b_1-1 are very small numbers. The strain-ellipse has the equation

$$a^2x^2+b^2y^2=r^2.$$

By the linear deformation certain stresses are produced which according to Clebsch [1] may be expressed in terms of the strains $a-1$ and b_1-1. In our special case there are no shearing stresses along the x- and y-axes, so that in the formulas [2]

$$N_1=\lambda\theta+2\mu a, \quad N_2=\lambda\theta+2\mu b$$

we have $\theta\equiv a+b_1-2$, $\lambda=\mu$ for a perfect solid, $a\equiv a-1$, $b=b_1-1$, $N_1=n_a$, $N_2=n_b$; hence

$$n_a=\lambda(3a+b_1-4),$$

$$n_b=\lambda(a+3b_1-4),$$

and the equation of the stress-ellipse

$$\frac{x^2}{\lambda^2(3a+b_1-4)^2}+\frac{y^2}{\lambda^2(a+3b_1-4)^2}=1.$$

§ 59. The Rectangular Pair of the Involution of Stresses in Nature.

In the sections corresponding to the rectangular pair of the involution only normal stresses are acting and these represent the maximal and minimal normal stresses. If at the point considered we advance an infinitesimal amount in the direction of one of the conjugate rectangular sections, for instance that for which the normal stress is a maximum, we can at this place, infinitely close to the first, again construct or calculate the two rectangular pairs of the involution. On the section for which

[1] *Theorie der Elasticität fester Körper*, Leipzig, 1862.

[2] MINCHIN: *Treatise on Statics*, Vol. II, p. 435, fourth edition.

the normal stress is a maximum we can again advance an infinitesimal distance and then construct the two conjugate normal sections, etc. In this manner a curve is obtained along which the normal stresses have their maximal values. If these stresses are tensions which are greater than the elastic limit of the material, then the material will rupture along this curve (maximal tension curve). In a similar way a curve may be drawn through the point along which the normal stresses have their minimal values. If the involution is hyperbolic, this curve is a maximal compression curve, since the stresses along this curve are maximal with reference to the compressive stresses.

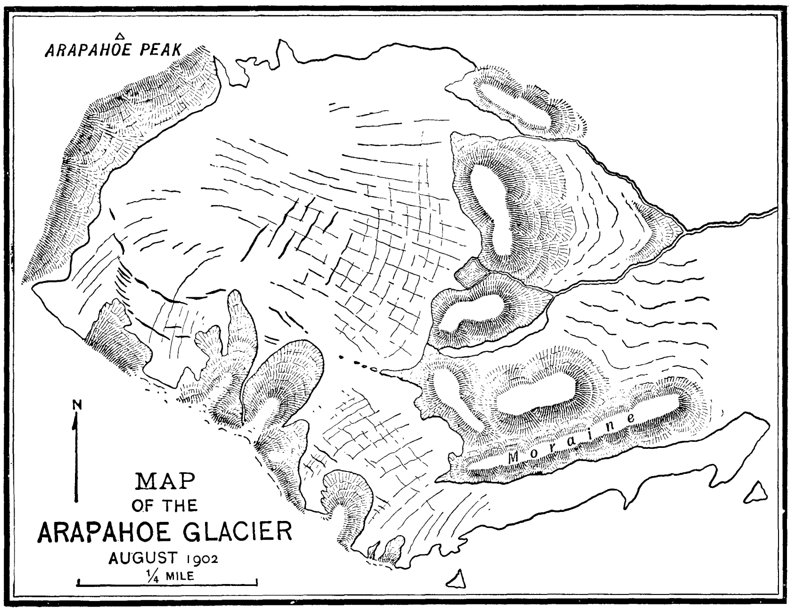


FIG. 102.

This case of a hyperbolic involution is shown in Fig. 102, which shows the crevasses of Arapahoe glacier.[1] The stream-

[1] From a drawing by Professor N. M. FENNEMAN in an article: *The Arapahoe Glacier in 1902*, Journal of Geology, Vol. X, p. 841.

lines represent the curves of maximal compression, while the crevasses cutting the stream-lines orthogonally represent the maximal tension curves. The case of an elliptic involution where there are only tensions is illustrated by the cracks which form on a heavily varnished surface or in a mud-bed which is drying up. In this case only tensile normal stresses act on the rectangular pair. One is a maximum, the other a minimum. We should therefore expect that the maximal tension curves would form a system of more or less parallel curves. This, however, does not occur, as is seen from Fig. 103, in which the

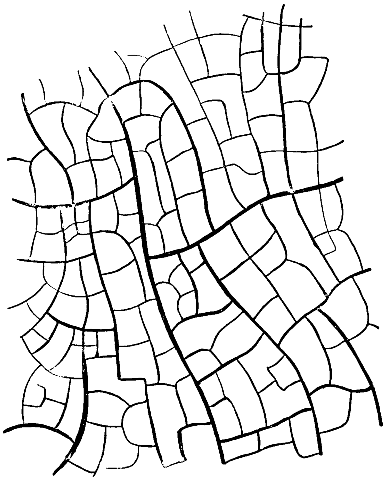


FIG. 103.

cracks intersect or meet orthogonally. This can be explained in the following manner: After a crack has formed, the maximal stress and strain normal to the crack has been relieved, so that the former minimal normal tension along the crack now becomes the maximum. Hence the next rupture will be orthogonal to the first crack.[1]

[1] See my article on this subject in the American Mathematical Monthly, Vol. VII, pp. 134, 135. Further examples may be found in Ritters *Graphische Statik*, Vol. I, pp. 128–134.

Realization of Collineations by Linkages.

§ 60. Introductory Remarks.

We have seen that collineations may be produced analytically and synthetically by different methods. In what follows a number of linkages will be described by which collineations may be realized kinematically. Linkages, like pantographs, translators, etc., devised for some special purpose have been known for a long time. The history of linkages in connection with the theory of geometrical transformations, however, dates back to the year 1864, when Peaucellier found a rigorous solution for the problem to describe a straight line by a link-motion.[1] Since that time a number of geometers, among whom the English Sylvester, Hart, Roberts, Cayley, and Kempe occupy the foremost places, have made a systematic study of linkages and their geometric properties and have found a great number of important results. Among these investigations probably the most interesting are those of Kempe and Koenigs. The first proved the theorem that it is always possible to find a linkage so that one of its points describes any given algebraic curve. Koenigs generalized this by proving that every algebraic surface and curve may be described by a linkage. As a result of these interesting theorems it is not difficult to prove that any algebraic transformation between any number of variables may be realized by linkages.[2] The difficulty lies in the actual construction of such linkages. Recently Koenigs has invented a linkage which realizes a general projective transformation in a plane.[3]

---

[1] Nouvelles Annales de Mathématiques, 2d series, Vol. III (1864), p. 144.

[2] Koenigs: *Leçons de Cinématique* (1897), pp. 262–307. See also Transactions of the American Mathematical Society, Vol. III, pp. 493–498 (Oct. 1902), where the author proves that *any number of algebraic relations between n complex variables may be realized by a plane linkage.*

[3] Comptes Rendus, Vol. CXXXI, p. 1179. The different cases of collineation were worked out by Hermann Emch in his *Master's Thesis* at the University of Colorado, 1902.

To have a definite idea about the character of the plane linkages to be considered I set down Koenigs's definition:

A plane linkage (système articulé, Gelenkwerk) is a combination of plates or plane figures subject to remain in one and the same plane, among which a certain number are connected to each other by hinges or pivots perpendicular to the common plane.

In this definition it is assumed that the links move by each other without interference, which means that the links, considered as material, lie in a series of close parallel planes.

Every linkage is constructed in such a manner that one of its pivots is fixed and represents the origin O, while others represent the algebraically related variables. The points of the linkages will always be designated by the same letters as the corresponding variables.

Two or more linkages each involving two variables may be combined in the following manner: Suppose L, L_1, L_2, . . . , L_n are linkages realizing the transformations

$$u=f(u_1), \quad u_1=f_1(u_2), \quad \dots, \quad u_{n-1}=f_{n-1}(u_n), \quad u_n=f_n(z).$$

Let the origins of all these linkages coincide; attach the pivot u_n of L_n to the pivot u_n of L_{n-1}; attach the pivot u_{n-1} of L_{n-1} to u_{n-1} of L_{n-2}, and so forth; finally the pivot u_1 of L_1 to u_1 of L. Then the point u of L evidently realizes the compound transformation

$$u=f\{f_1[f_2 \dots f_{n-1}(f_n(z))]\}=F(z).$$

Linkages involving more than two variables may be similarly combined.

The range of effectiveness of a linkage is, of course, limited to a certain finite portion of the plane. This range, although in some cases small, always exists.

§ 61. Analytic Formulation of the Problem.

We shall consider only the most important cases of collineation. A great number of special cases will be left as exercises for the student.

The most important cases are the *linear transformation* and *perspective*. A linear transformation

$$(1) \qquad \begin{cases} x_1 = ax + by + c, \\ y_1 = dx + ey + f \end{cases}$$

may be considered as made up of four subgroups: (1) the two-termed *translation*, (2) the one-termed *rotation*, (3) the two-termed *dilation*,[1] (4) the one-termed *elation*. By a translation (p, q) and a rotation (ϕ) the point (x, y) is changed into (x', y').

$$(2) \qquad \begin{cases} x' = x \cos \phi - y \sin \phi + p, \\ y' = x \sin \phi + y \cos \phi + q. \end{cases}$$

A dilation

$$(3) \qquad \begin{cases} x'' = \alpha x', \\ y'' = \beta y' \end{cases}$$

changes (x', y') into (x'', y''):

$$(4) \qquad \begin{cases} x'' = \alpha x \cos \phi - \alpha y \sin \phi + \alpha p, \\ y'' = \beta x \sin \phi + \beta y \cos \phi + \beta q. \end{cases}$$

Finally by the elation

$$(5) \qquad \begin{cases} x_1 = x'' + \lambda y'', \\ y_1 = y'' \end{cases}$$

we get

$$(6) \begin{cases} x_1 = (\alpha \cos \phi + \lambda\beta \sin \phi)x + (\lambda\beta \cos \phi - \alpha \sin \phi)y + \alpha p + \lambda\beta q, \\ y_1 = \beta \sin \phi \cdot x + \beta \cos \phi \cdot y + \beta q, \end{cases}$$

[1] Term used by S. Lie, loc. cit. It is equivalent with *dilatation*, p. 60.

which by properly choosing α, β, λ, ϕ, p, q, the six parameters, may represent any linear transformation of (x, y) into (x_1, y_1). To prove this let

$$\begin{aligned} \alpha \cos \phi + \lambda\beta \sin \phi &= a, \\ \lambda\beta \cos \phi - \alpha \sin \phi &= b, \\ \alpha p + \lambda\beta q &= c, \\ \beta \sin \phi &= d, \\ \beta \cos \phi &= e, \\ \beta q &= f, \end{aligned}$$

which represent six equations with six unknown quantities α, β, λ, ϕ, p, q. Solving, we get

$$\alpha = \frac{ae - bd}{d^2 + e^2}, \quad \beta = \sqrt{d^2 + e^2}, \quad \lambda = \frac{ad + be}{d^2 + e^2},$$

$$\phi = \text{arc tan } \frac{d}{e}, \quad p = \frac{c(d^2 + e^2) - f(ad + be)}{ae - bd}, \quad q = \frac{f}{\sqrt{d^2 + e^2}}.$$

Substituting these values in (2), (3), (5), we obtain a translation with rotation, a dilation, and an elation which in succession transform (x, y) into (x', y'), (x', y') into (x'', y''), and finally (x'', y'') into (x_1, y_1) in such a manner that (x_1, y_1) is connected to (x, y) by the linear transformation

$$(7) \qquad \begin{cases} x_1 = ax + by + c, \\ y_1 = dx + ey + f. \end{cases}$$

Applying to (x_1, y_1) the perspective

$$(8) \qquad \begin{cases} x' = \dfrac{x_1}{d_1 x_1 + e_1 y_1 + f_1}, \\ y' = \dfrac{y_1}{d_1 x_1 + e_1 y_1 + f_1}, \end{cases}$$

we have

$$
(9) \qquad \begin{cases} x' = \dfrac{ax+by+c}{(d_1a+e_1d)x+(d_1b+e_1e)y+f_1}, \\ y' = \dfrac{dx+ey+f}{(d_1a+e_1d)x+(d_1b+e_1e)y+f_1}, \end{cases}
$$

which may represent any projective transformation. The linear transformation is a six-termed and the perspective a three-termed group, so that their combination (9) contains nine parameters, although the general projective group is eight-termed. This is due to the fact that both the linear transformation and the perspective contain the one-termed group of similitudes as a subgroup. These considerations, which may be found in a little different form in § 19, have been repeated here for a clearer understanding of what follows.

We shall now proceed to describe linkages realizing the transformations in question. Theoretically only such linkages should be admissible in which a link joins two and only two points. In other words, no three points in a straight line should be admitted *a priori*. It is, however, very useful for practical purposes to make this last assumption. For some transformations we shall construct more than one linkage in order to show the advantage which one or the other may have.

Combining the linkages involved in the linear transformation and in perspective according to the scheme explained in the last part of § 60, a compound linkage for a general collineation is obtained.

§ 62. Peaucellier's Inversor.

In our particular investigation of link-motions the problem to draw a straight line theoretically correct is of the greatest importance. This can be done by Peaucellier's inversor (loc. cit.) or by Hart's linkage (Koenigs's Cinématique, p. 267). Peaucellier's inversor is of greater principal value and will be described here.

It consists in the first place of a rhombus $ABPP'$ and two equal links AO and BO. In all these points the links are joined

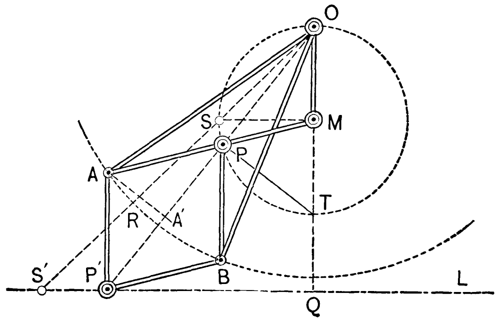


FIG. 104.

by pivots, Fig. 104. Designating $OA=OB$ by a, $AP=PB=BP'=P'A=b$, $AA'=c$, we have $OA'=\frac{1}{2}(OP+OP')$, $A'P=\frac{1}{2}(OP'-OP)$; $\frac{1}{4}(OP+OP')^2=a^2-c^2$; $c^2=b^2-\frac{1}{4}(OP'-OP)^2$, or $\frac{1}{4}(OP+OP')^2-\frac{1}{4}(OP'-OP)^2=a^2-b^2$, or finally

$$OP\cdot OP'=a^2-b^2.$$

Hence, if O is kept fixed, the points P and P' are inverse with respect to a circle having O as a center and $\sqrt{a^2-b^2}$ as a radius. If now P describes a circle with M as a center and $OM=r$ as a radius, we have $OP\cdot OP'=OT\cdot OQ$, or $OP/OT=OQ/OP'$; consequently $\triangle OPT \backsim \triangle OQP'$; and since $\angle OPT=90°$, also $\angle OQP'$ will be a right angle. *Hence, when P describes said circle, P' will describe a straight line perpendicular to the direction of OM.*

For the limiting position $OSRS'$ of the inversor we have

$$QS'=\sqrt{OS'^2-OQ^2},$$

or, since

$$OS'^2=(a+b)^2 \quad \text{and} \quad OQ=\frac{a^2-b^2}{2r},$$

$$QS'=\frac{1}{2r}\sqrt{(a+b)^2 4r^2-(a^2-b^2)^2}.$$

Of course the lengths a, b, r must be chosen in such a manner that the linkage is movable. The conditions are $r > \frac{a-b}{2}$, following from $(a+b)^2 4r^2 - (a^2-b^2)^2 > 0$, and, for the case that the straight line shall not cut the circle of inversion $r \leqq \frac{1}{2}\sqrt{a^2-b^2}$.

Ex. 1. Show that when M, P, and A are in a straight line, $AP' \perp S'Q$.

Ex. 2. If the whole linkage is in a vertical plane and OM vertical, the linkage remains in equilibrium under the action of any weight suspended at P'.

Ex. 3. If a and b are given, what value must r have to make QS' a maximum?

§ 63. Pantographs.

1. *Inversor Pantograph.*

By means of Peaucellier's cell $ABPQO$, a part of the inversor,

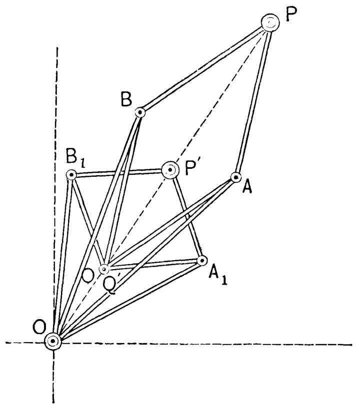


Fig. 105.

we can locate for every point P a point Q, so that $OP \cdot OQ = a^2 - b^2$. Applying another cell, $A_1B_1Q'P'O_1$, for which $O_1Q' \cdot O_1P' = a_1{}^2 - b_1{}^2$,

and letting O_1 coincide with O, and Q with Q', Fig. 105, then $O_1Q'=OQ'$, and by division

$$\frac{OP}{OP'}=\frac{a^2-b^2}{a_1^{\,2}-b_1^{\,2}}.$$

Hence, when P describes a figure, P_1 will describe a similar figure. Choosing O as the origin of Cartesian coordinates and designating the constant ratio $\dfrac{a^2-b^2}{a_1^{\,2}-b_1^{\,2}}$ by κ, then when O is fixed, the linkage of Fig. 105 will realize the transformation of $P(x, y)$ into $P'(x', y')$:

$$x'=\kappa x, \quad y'=\kappa y.$$

2. *Sylvester's Pantograph.*[1]

Take any two similar triangles, Fig. 106, OAA' and APB pivoted at A, with $\angle A'OA=\angle BAP$ and $\angle A'AO=\angle BPA$.

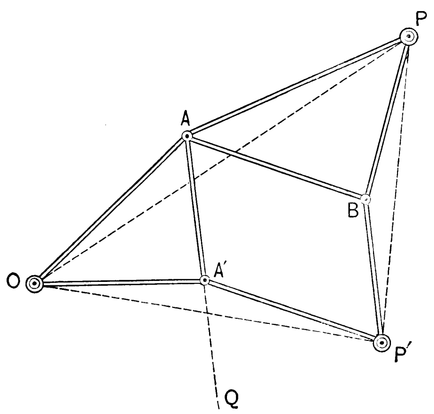


Fig. 106.

Now $\angle QA'P'=\angle A'AB$, $\angle QA'O=\angle A'OA+\angle A'AO$; hence $\angle QA'P'+\angle QA'O=\angle A'AB+\angle BAP+\angle A'AO$, or $\angle OA'P'=\angle OAP$. But there is also

$$\frac{OA}{OA'}=\frac{AP}{AB}=\frac{AP}{A'P'},$$

[1] *Nature*, 1875, p. 168.

hence $\triangle OA'P' \backsim \triangle OAP$. From this $\frac{OA}{OA'} = \frac{OP}{OP'}$ and $\angle AOP = \angle A'OP'$, hence also $\angle POP' = \angle AOA'$ and $\triangle POP' \backsim \triangle AOA'$. Consequently when P describes a figure and O remains fixed, P' will describe a similar figure. The ratio of similitude between the figures traced by P and P' is OA/OA'. Turning the figure traced by P' negatively through an angle $= \angle A'OA$ it will come similarly situated with the figure traced by P with respect to the center O. Designating by ϕ the angle $A'OA$, and by ρ the ratio $\frac{OA'}{OA}$, Sylvester's pantograph will realize the combined groups of rotation and similitude between $P(x, y)$ and $P'(x', y')$:

$$x' = \rho(x \cos\phi - y \sin\phi),$$
$$y' = \rho(x \sin\phi + y \cos\phi).$$

This becomes a pure rotation when $\rho = 1$; i.e., $OA = OA'$.

The arrangement of this linkage is a little different from Sylvester's original pantograph, but does not essentially differ from it.

3. *P. Scheiner's Pantograph* (1631).

Scheiner's or the ordinary pantograph appears in the market under many different forms. One of the simplest is illustrated in Fig. 107. Two equal sets of links PQ, PR and CP', CO

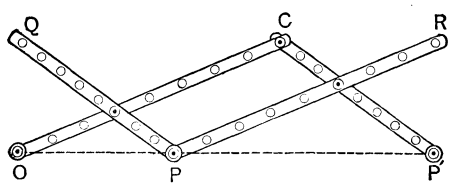


FIG. 107.

pivoted at P and C are placed in such a manner that P is in a straight line with O and P', and $PQ \parallel CP'$, $PR \parallel CO$. In this position pivots are also placed where PR and CP', and PQ and CO meet. From the figure it appears at once that when O is fixed and P describes a figure, then P' will describe a figure

similar to and similarly situated with the first. The linear ratio between the two figures is OP/OP'. To make different values for this ratio possible the links may be divided into equal parts, as shown in Fig. 107. Wishing to enlarge a figure in the linear ratio 4 : 3, set the pivots where PR and CP', and PQ and CO meet at the marks 6, so that $OP'/OP=8:6=4:3$. In a similar manner arrangements for any other ratio may be made by properly dividing the links.

Although Scheiner's pantograph is the simplest of all pantographs and consequently exclusively used for practical purposes, it has the theoretical disadvantage of not being a pure linkage. Indeed, in Fig. 107 it is assumed that a link joins three given points in a straight line. The first two pantographs described are pure linkages.

§ 64. Rotator and its Combinations.[1]

1. To realize a rotation through an angle ϕ of a point $P(x, y)$ into $P'(x', y')$, so that

$$x'=x\cos\phi-y\sin\phi,$$
$$y'=x\sin\phi+y\cos\phi,$$

Sylvester's pantograph in the case $\rho=1$ may be used. Another linkage for the same purpose, Fig. 108, is obtained by taking two isosceles triangles OAC and OBD pivoted at O, the coordinate-origin, with $\angle AOC=\angle BOD=\phi$ and $AO=CO=BO=DO$. Attaching the links $PB=PC$, $P'A=P'D$, pivoted at P and P' respectively, and all equal to AO, two equal rhombs $OBPC$ and $OAP'D$ are obtained. Hence $\angle AOP=\angle BOP'$, $\angle POB=\angle P'OA$, and $\angle BOP'+\angle POB=\angle AOP+\angle COP$, or

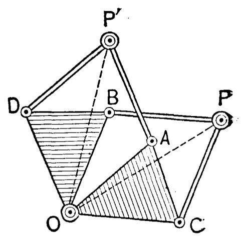


FIG. 108.

$$\angle POP'=\angle AOC=\phi.$$

[1] A paper on this linkage and its applications was presented to the Am. Math. Soc. in Chicago, Sept. 1902, and was published in Vol. I of *The University of Colorado Studies,* April 1903. See also *Transactions of the Am. Math. Soc.* Vol. III, No. 4, pp. 493-498, Oct. 1902.

Furthermore, $P'O = PO$. The linkage of Fig. 108 can therefore be used to perform the proposed rotation.

2. The foregoing linkage may be used for various purposes. In the first place when O is not fixed, we have in the three pivots P, O, P' a variable isosceles triangle which in all its deformations remains similar to some original size. When O is fixed and P describes a straight line or a circle, P' also describes a straight line or a circle respectively. Making $\phi = 90°$, and taking two equal rotators with the points P and P' attached, a variable square is obtained.

§ 65. Translators.

One of the simplest devices for translation is that of Kempe.[1] It consists of three parallel equal links AD, BC, PP' which are connected by $CD = \| AB$ and $CP' = \| BP$. Letting A coincide with the coordinate-origin and designating the coordinates of D by a and b, then for the coordinates (x, y) of P and (x', y') of P' we have from Fig. 109

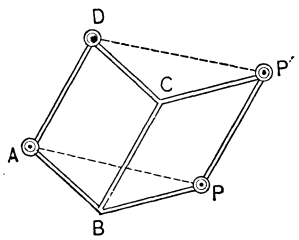


FIG. 109.

$$x' = x + a,$$
$$y' = y + b.$$

A translator which is more general is obtained from a linkage which was originally invented to perform the addition of any complex variables.[2] It consists of 12 links, Fig. 110, of which $OF = \| CE = \| BD$; $FP' = \| EG = \| DP$; $OC = \| FE = \| P'G$; $CB = \| ED = \| GP$. It is evident that $OBPP'$ will always be a parallelogram no matter how the linkage may be deformed. Hence, keeping B and O fixed, P' represents in every position of the linkage a translation of P equal to BO and in the direction of BO.

---

[1] *How to Draw a Straight Line.*

[2] *Transactions*, loc. cit.

§ 66. Linear Transformation.

1. By a combination of rotator and translator it is possible to realize a general motion in the plane. According to § 61

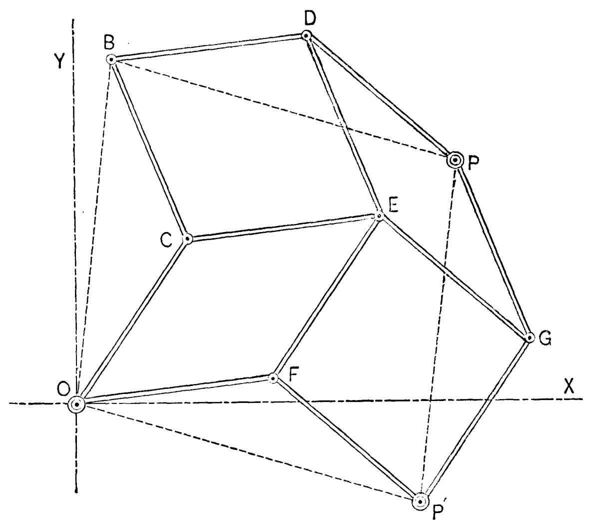


FIG. 110.

the next group in making up a linear transformation is the dilation

$$x'' = \alpha x'.$$
$$y'' = \beta y'.$$

The linkage for this transformation is shown in Fig. 111.[1] Let $OA = x'$. By a Scheiner pantograph (which we choose for the sake of simplicity), in which $OB/OA = \alpha$ and consisting of the links OC, CB, DA and AE, a point B is realized for which $BO = x'' = \alpha x'$. One of the points A and B is kept on the x-axis by a Peaucellier inversor. To A and B attach a translator $ABIP'VLMNH$. Produce the link NB arbitrarily to R and

[1] This is essentially the arrangement of HERMANN EMCH in his thesis, loc. cit.

complete the rhombus $BRST$. To BR and BT attach at B the equal right-angled triangles RBF and TBG, so that $BF=BG$. Complete the rhombus $BFGP''$. From the figure follows easily that $\angle FBG = \angle RBT$ and that $P''B \perp BS$, $\perp OB$.

Now use FB and FP'' as links of a second Scheiner pantograph, and attach the links IK and IU in such a manner that

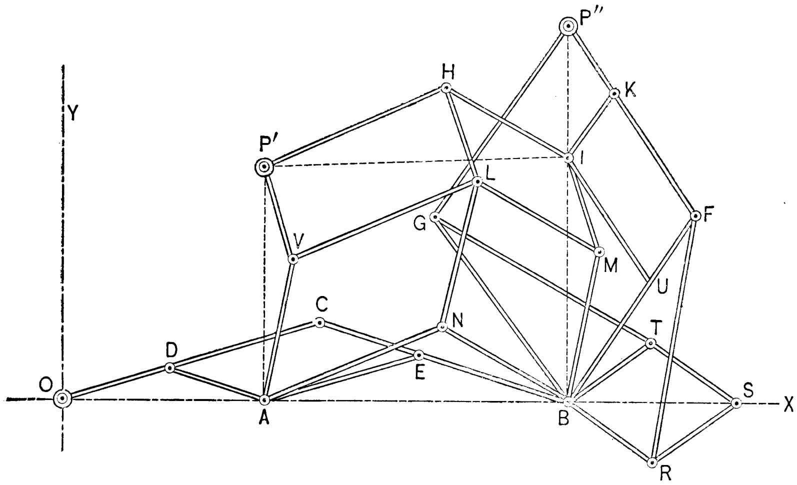


FIG. 111.

$BF : IK = P''F : IU = BP'' : IP''$, and $P''B/BI = \beta$. The point I is now collinear with P'' and B, and as $P'A \parallel IB$ it follows that $P'A \perp OA$. Making $P'A = y'$, we find $IB = P'A = y'$; $P''B = \beta \cdot IB = \beta y'$. The coordinates of P'' are therefore

$$BO = x'' = \alpha x',$$

$$P''B = y'' = \beta y',$$

and we have constructed a linkage realizing dilation.

2. The last group to be considered is the one-termed elation. Take two rhombs $AEP''F$ and $ACBD$ with the common joint or pivot A; join E and C, and F and D by two equal links $EC = FD$,

so that $\angle EAC = \angle FAD = \frac{\pi}{2}$. From plane geometry there follows easily $\angle EAF = \angle CAD$; i.e., that the two rhombs are similar; further, that $P''A \perp AB$, no matter how the linkage may be distorted. *This linkage realizes, therefore, a variable right triangle $P''AB$ whose angles are constant.* Joining in a symmetrical manner the rhomb $BHP_1G = AEP''F$ to the previous linkage ($CG = CE$, $HD = FD$), *a variable rectangle $ABP''P_1$ is obtained whose sides have a constant ratio*, Fig. 112.

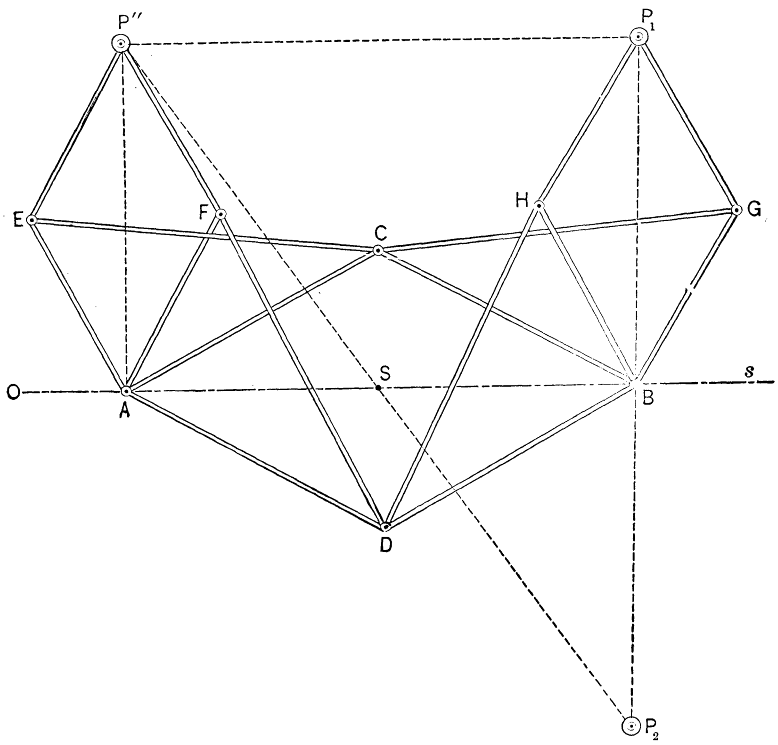


FIG. 112.

This linkage may be used to solve mechanically two interesting cases of collineations in a plane. If by two Peaucellier Inversors the points A and B are forced to describe the same

straight line s, in which an arbitrary point is taken as the origin of a Cartesian coordinate system, and s itself is assumed as the x-axis, we have, since $\dfrac{AB}{AP''}=m$ (constant), for the coordinates x_1, y_1 of P_1 in terms of those of $P''(x'', y'')$:

$$x_1 = x'' + my'',$$

$$y_1 = y'',$$

which represents the required elation.

If the rhombus $BGHP_1$ and the links CG and DH are attached below s, so that the point P_1 will fall on P_2, then the coordinates of P_2 are

$$x_2 = x'' + my'',$$

$$y_2 = -y'',$$

which represents oblique axial symmetry. Combining the linkages for rotation, translation, dilation, and elation as explained in § 60, a linkage for the linear transformation is obtained.

Ex. 1. Construct a linkage for the transformation (oblique axial symmetry):

$$x' = x + my,$$

$$y' = -y.$$

Ex. 2. Construct a linkage for the transformation (orthogonal axial symmetry):

$$x' = x,$$

$$y' = -y.$$

Ex. 3. Construct a linkage for the special dilation:

$$x' = \alpha x,$$

$$y' = y.$$

Ex. 4. Construct a linkage for the central symmetry:

$$x' = -x,$$

$$y' = -y.$$

Ex. 5. Draw the combined linkage for a general linear transformation.

Ex. 6. Determine the ranges, or limits of the areas, covered by Sylvester's pantograph, the rotator, the translators, and Scheiner's pantograph as used in the linkage for dilation.

§ 67. Perspective.

1. Mechanisms by which the perspective of any plane figure may be drawn are known in various forms. One that is in practical use is the "perspectivograph" invented by H. RITTER.[1] In this mechanism pivots are kept on given straight lines by grooves so that it combines link- and sliding-motions. Another "perspectivograph" in which two ellipses are used and which also combines link- and sliding-motions was described by the author some years ago.[2] Probably the most important linkage-realizing perspective has been invented by KOENIGS,[3] and, as it does not use slide-motion, will be described here. We must, however, first describe KEMPE's reversor which Koenigs uses as an auxiliary linkage.

2. *Kempe's Reversor.*

In Fig. 113 consider the linkage $OBCD$ in which OB and CD are equal and cross each other, and also $OD = BC$. Designating the variable point of intersection of OB and CD by X, this linkage, which is called counter-parallelogram, has the property that for any deformation $\triangle BOD = \triangle DCB$; $\triangle OXD = \triangle CXB$. On DC choose a pivot E in such a manner that $DE:DO = DO:DC$, so that the triangles OCD and EOD are similar. Then with OD

---

[1] See *Geometrische Transformationen* by Dr. K. DOEHLEMANN, pp. 199–204, Leipzig, 1902.

[2] *The Industrialist*, Vol. XXV, pp. 237–240, Manhattan, Kansas, 1899.

[3] *Comptes Rendus*, Vol. CXXXI, p. 1179.

and ED as given links complete the counter-parallelogram $ODEF$, in which $FE=DO$, $FO=DE$. Thus $\triangle EOD = \triangle OEF$ and similar to $\triangle OCD = \triangle COB$. Hence, in every deformation, the

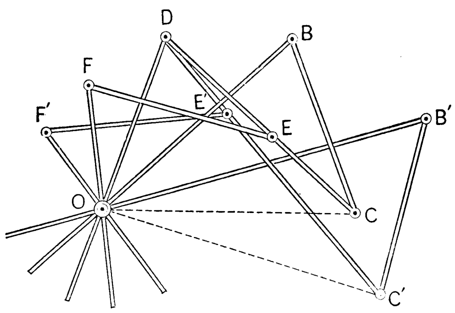


FIG. 113.

counter-parallelograms $OBCD$ and $ODEF$ are similar, and as $\triangle BOD = \triangle DCB$, it also follows that $\triangle DOF = \triangle FED$. By means of this reversor it is possible to keep two links BO and FO at equal angles from a given link DO. By a similar construction two other links, $B'O$ and $F'O$, symmetrical to DO may be attached, and it is clear that their motion is otherwise independent of that of BO and FO. We have therefore a linkage in which in every deformation

$$\angle BOB' = \angle FOF'.$$

Kempe's reversor may be extended to realize any number of equal angles, $\angle BOD = \angle DOF = \angle FOH = \dots$ For the details of this we refer to Koenigs's Kinematics, loc. cit.

KOENIGS' PERSPECTIVOGRAPH.

3. Introducing polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, in the formulas for a perspective transformation

$$(1) \qquad \begin{cases} x' = \dfrac{x}{dx+ey+f}, \\ y' = \dfrac{y}{dx+ey+f}, \end{cases}$$

we get, since $\theta' = \theta = \text{arc tan}\left(\frac{y}{x}\right) = \text{arc tan}\left(\frac{y'}{x'}\right)$,

$$(2) \qquad r' = \frac{r}{r(d \cos \theta + e \sin \theta) + f},$$

and

$$(3) \qquad \frac{1}{r'} = d \cos \theta + e \sin \theta + \frac{f}{r}.$$

Putting $d = -\frac{1}{a} \sin \phi$, $e = -\frac{1}{a} \cos \phi$, so that $a = \frac{1}{\sqrt{d^2+e^2}}$ and $\phi = \tan^{-1}\left(\frac{d}{e}\right)$, (3) becomes

$$(4) \qquad \frac{1}{r'} - \frac{f}{r} = -\frac{1}{a} \sin (\theta + \phi).$$

Now take two Peaucellier inversors $OABQP$ and $OCDQ'P'$, Fig. 114, and by means of Kempe's reversor as described in

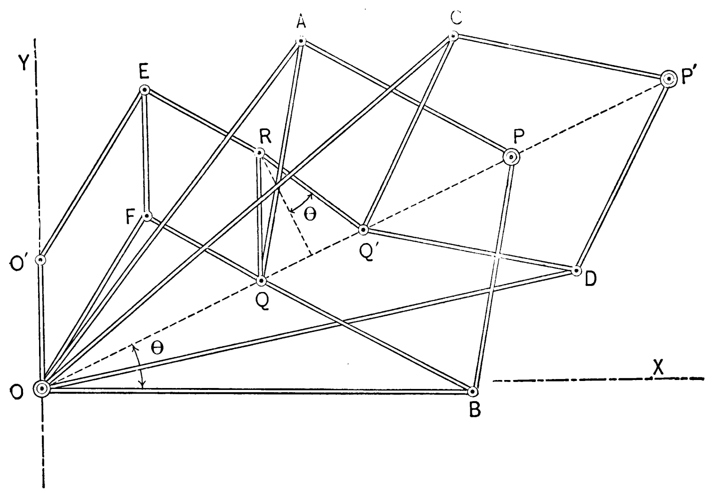


Fig. 114.

Fig. 113 keep $\angle AOC = \angle BOD$. This can be done by properly choosing B', F', B, F of Fig. 113 on BO, AO, DO, CO of Fig. 114,

respectively. Let $OP=r$, $OP'=r'$; μ and μ' the squares of the radii of the circles of inversion of the two inversors; $OQ=\rho$, $OQ'=\rho'$. Now $OP\cdot OQ=\mu$, or $r\cdot\rho=\mu$, hence $\frac{1}{r}=\frac{\rho}{\mu}$; $OP'\cdot OQ'=\mu'$, or $r'\cdot\rho'=\mu'$, hence $\frac{1}{r'}=\frac{\rho'}{\mu'}$. So far, no particular values are assigned to μ and μ', so that we can choose $\mu=j\cdot\mu'$; $\frac{j}{r}=\frac{\rho}{\mu'}$. Equation (4) now becomes

$$(5) \qquad \rho'-\rho=-\frac{\mu'}{a}\sin(\theta+\phi).$$

To the two inversors attach a Kempe translator $OO'EFRQ$, where O' is on the y-axis, and join R to Q' by the link $RQ'=RQ$. Let $\angle POX=\theta+\phi$; then $\angle QRQ'=2(\theta+\phi)$, and $QQ'=\rho'-\rho=2RQ\cdot\sin(\theta+\phi)$. Hence, taking $OO'=RQ=-\frac{1}{2}\frac{\mu'}{a}=\frac{\mu'}{2}\sqrt{d^2+e^2}$, the points P' and P realize the proposed perspective transformation, since the linkage satisfies all conditions of equations (1) or (2), or their equivalent (5).

Ex. 1. Modify the linkage so that P and P' describe similar figures.

Ex. 2. Investigate the cases $j=1$, and $j=0$.

INDEX.